BEC-T_c: Second-Order Shift for Dilute Bose Gas

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PROBLEM

determine numerical values of $\bar{c}_1$, $\bar{c}_2'$, $\bar{c}_2''$, and $\breve{c}_1$, $\breve{c}_2'$, $\breve{c}_2''$ in shifts

hom. gas [7]: $$\frac{\Delta T_c}{T_0} = \bar{c}_1 a n^{1/3} + [\bar{c}_2' \ln(a n^{1/3}) + \bar{c}_2''](a n^{1/3})^2 + \cdots, \qquad T_0 = \frac{2\pi}{m}\left[\frac{n}{\zeta(\frac{3}{2})}\right]^{2/3},$$

contributions	$\bar{c}_1$	$\bar{c}_2'$	$\bar{c}_2''$
perturbative	no	yes	yes
non-perturbative	yes	no	yes

wide trap [6]: $$\frac{\Delta T_c}{T_0} = \breve{c}_1 \frac{a}{\lambda_0} + \left(\breve{c}_2' \ln\frac{a}{\lambda_0} + \breve{c}_2''\right)\left(\frac{a}{\lambda_0}\right)^2 + \cdots, \qquad T_0 = \left(\frac{N_b}{\zeta(3)}\right)^{1/3} \frac{1}{m l_{\text{ho}}^2},$$

contributions	$\breve{c}_1$	$\breve{c}_2'$	$\breve{c}_2''$
perturbative	yes	yes	yes
non-perturbative	no	no	yes

a = scattering length, n = boson density, λ_0 = thermal wavelength at T_0, m = mass of bosons, $l_{\text{ho}} = \frac{1}{\sqrt{m(\omega_x\omega_y\omega_z)^{1/3}}}$ = size of unperturbed h.o. ground state

THEORY

non-perturbative contributions from 3-dim. critical field theory [6,7]

3+1-dim. field theory for grand-canonical ensemble of bosons in imaginary time:
$$S_{3+1} = \int_0^\beta d\tau \int d^3x \left[\psi^* \left(\frac{\partial}{\partial\tau} - \frac{1}{2m}\nabla^2 - \mu\right)\psi + \frac{2\pi a}{m}(\psi^*\psi)^2\right]$$

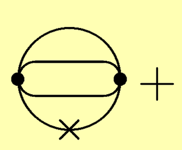
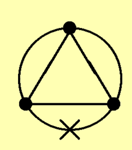
↓

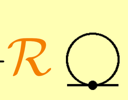
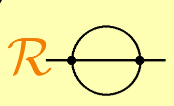
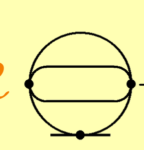
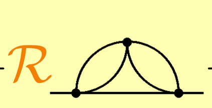
match to 3-dim. field theory of zero Matsubara modes:
$$S_3 = \beta \int d^3x \left[\psi_0^* \left(-\frac{Z_\psi}{2m}\nabla^2 - \mu_3\right)\psi_0 + Z_a \frac{2\pi a}{m}(\psi_0^*\psi_0)^2 + f_3 + \cdots\right]$$

↓

translate to standard formulation of 3-dim. field theory:
$$S_3 = \int d^3x \left[\frac{1}{2}|\nabla\phi|^2 + \frac{r_B}{2}\phi^2 + \frac{u}{24}(\phi^2)^2\right]$$
with
$$\psi_0 = \sqrt{mT/Z_\psi}\,(\phi_1 + i\phi_2), \quad r_B = -2m\mu_3/Z_\psi, \quad u = 48\pi a m T Z_a/Z_\psi^2$$
generalize: $\phi = (\phi_1, \ldots, \phi_N)$

S_3 superrenormalizable; ren. scheme: $r \equiv r_{\text{B}} - \Sigma(0)$ ⇝ full prop.: $G(p) = \frac{1}{p^2 + r - [\Sigma(p) - \Sigma(0)]}$; free prop.: $G_0(p) = \frac{1}{p^2 + r}$; critical limit: $r \to 0$

need for $\bar{c}_1$, $\bar{c}_2''$:
$$\Delta\langle\phi^2\rangle\big|_{\text{crit.}} \equiv \lim_{r\to0}\left(\langle\phi^2\rangle - \langle\phi^2\rangle\big|_{u=0}\right) = \lim_{r\to0} N\int_p [G(p) - G_0(p)] = \lim_{r\to0}\Big(\mathcal{R}$$  $$+\mathcal{R}$$  $$+\cdots\Big) = Nu \lim_{r\to0}\sum_{l=1}^{\infty} a_l \Big(\underbrace{\frac{Nu}{4\pi\sqrt{r}}}_{\equiv \bar{u}_r}\Big)^{l-2} \equiv Nu\kappa_N$$

need for $\bar{c}_2''$, $\breve{c}_2''$:
$$r_{\overline{\text{MS}}}\big|_{\text{crit.}} \equiv \lim_{r\to0}\lim_{\epsilon\to0}\left(r_{\text{B}}(r) - \frac{(N+2)u^2}{36(4\pi)^2\epsilon}\right)\Bigg|_{\bar{\mu}=(N+2)u/12} = \lim_{r\to0}\lim_{\epsilon\to0}\left(r + \Sigma(0,u,r) - \frac{(N+2)u^2}{36(4\pi)^2\epsilon}\right)\Bigg|_{\bar{\mu}=(N+2)u/12}$$
$$= \lim_{r\to0}\lim_{\epsilon\to0}\Big[r + \mathcal{R}$$  $$+ \Big(\mathcal{R}$$  $$- \frac{(N+2)u^2}{36(4\pi)^2\epsilon}\Big) + \mathcal{R}$$  $$+ \mathcal{R}$$  $$+ \cdots\Big]\Bigg|_{\bar{\mu}=(N+2)u/12} = (N+2)u^2 \lim_{r\to0}\left[b_2' \ln\frac{(N+2)u}{\sqrt{r}} + \sum_{l=0}^{\infty} b_l \Big(\underbrace{\frac{(N+2)u}{\sqrt{r}}}_{\equiv \tilde{u}_r}\Big)^{l-2}\right] \equiv (N+2)u^2 R_N$$

$$\bar{c}_1 = -\frac{4(4\pi)^3}{\zeta(\frac{3}{2})^{4/3}}\kappa_2, \quad \bar{c}_2' = -\frac{16(4\pi)\zeta(\frac{1}{2})}{3\zeta(\frac{3}{2})^{5/3}} \approx 19.7518, \quad \bar{c}_2'' = \frac{16(4\pi)\zeta(\frac{1}{2})}{9\zeta(\frac{3}{2})^{5/3}}\ln[\zeta(\tfrac{3}{2})] + \frac{28(4\pi)^6}{\zeta(\frac{3}{2})^{8/3}}\kappa_2{}^2 + \frac{8(4\pi)}{3\zeta(\frac{3}{2})^{5/3}}\left\{\frac{\zeta(\frac{1}{2})^2\ln 2}{\sqrt{\pi}} + 2K_2 - \sqrt{\pi} - [\ln 2 + 3\ln(4\pi) + 1 - 36(4\pi)^2 R_2 - 24(4\pi)^2\kappa_2]\zeta(\tfrac{1}{2})\right\}$$

$$\breve{c}_1 = \frac{2}{3\zeta(3)}\left[\sum_{i,j=1}^{\infty}\frac{1}{i^{3/2}j^{3/2}(i+j)^{1/2}} - 2\zeta(2)\zeta(\tfrac{3}{2})\right] \approx -3.426032, \quad \breve{c}_2' = -\frac{8(4\pi)\zeta(2)}{3\zeta(3)} \approx -45.8566, \quad \breve{c}_2'' = C_2 - \frac{4(4\pi)\zeta(2)}{3\zeta(3)}[1 - \ln 2 + 3\ln(4\pi) - 4K_1 - 36(4\pi)^2 R_2]$$

METHOD

loop expansions for κ_N and R_N are IR divergent as $r \to 0$; solution: resum using Kleinert's Variational Perturbation Theory (VPT) [1,2]:

L loops: (i) replace $\overset{(\sim)}{u}_r{}^{l-2} \to \left(\frac{(t\overset{(\sim)}{u}_r)}{\{(\overset{(\sim)}{u}_r/\hat{u})^\omega + t[1-(\overset{(\sim)}{u}_r/\hat{u})^\omega]\}^{1/\omega}}\right)^{l-2}$, (ii) expand through t^{L-2}, then set $t = 1$, (iii) let $\overset{(\sim)}{u}_r \to \infty$ (in our case), (iv) optimize in $\hat{u}$ (PMS)

RESULTS

N	κ_N from MC	κ_N from 7-loop VPT
0	—	$-(3.66 \pm 0.39) \times 10^{-4}$ [10]
1	$-(4.94 \pm 0.41) \times 10^{-4}$ [8]	$-(4.86 \pm 0.45) \times 10^{-4}$ [9]
2	$-(5.85 \pm 0.23) \times 10^{-4}$ [4] $-(5.99 \pm 0.09) \times 10^{-4}$ [5]	$-(5.75 \pm 0.49) \times 10^{-4}$ [9]
3	—	$-(6.46 \pm 0.48) \times 10^{-4}$ [10]
4	$-(7.23 \pm 0.45) \times 10^{-4}$ [8]	$-(6.99 \pm 0.48) \times 10^{-4}$ [9]
∞	$-1/[6(4\pi)^2] \approx -1.05543 \times 10^{-3}$ [3]	

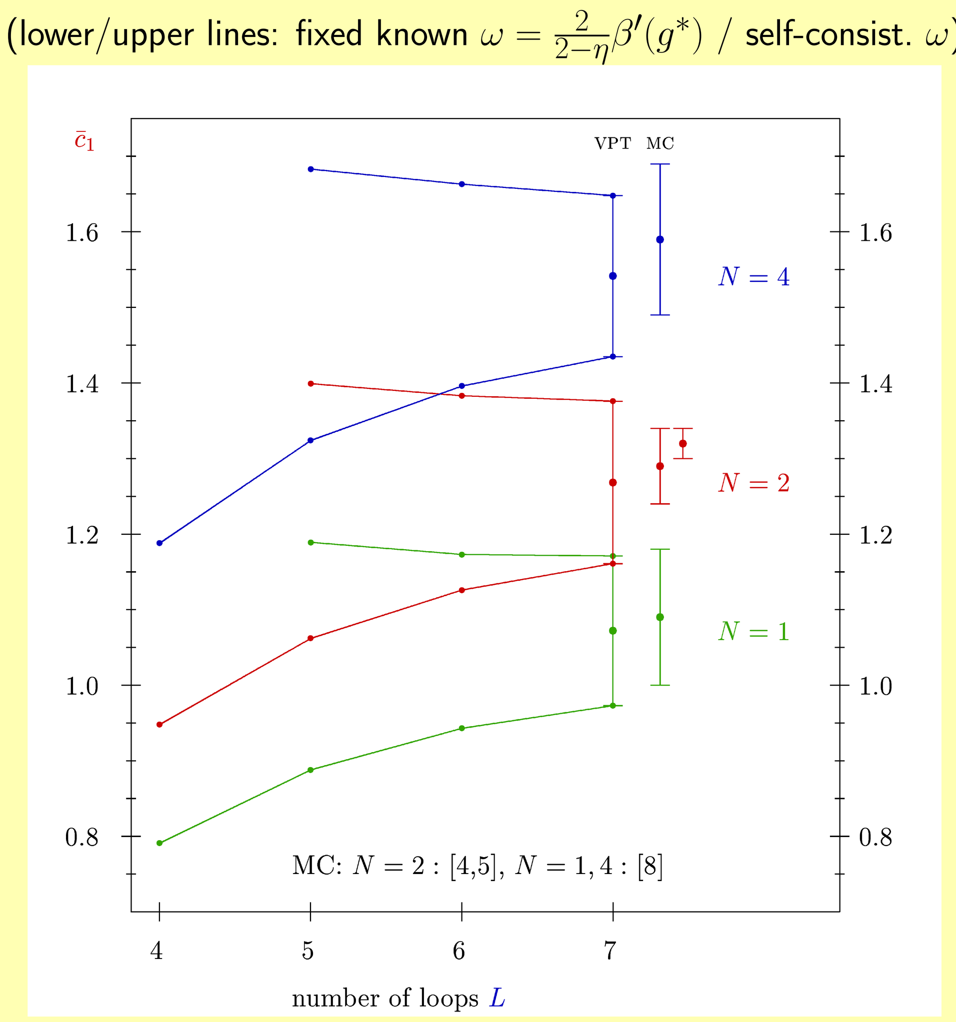
N	R_N from MC	R_N from 7-loop VPT
0	—	$(2.884 \pm 0.052) \times 10^{-4}$ [10]
1	$(4.071 \pm 0.016) \times 10^{-4}$ [8]	$(4.047 \pm 0.051) \times 10^{-4}$ [10]
2	$(4.8003 \pm 0.0053) \times 10^{-4}$ [5]	$(4.804 \pm 0.047) \times 10^{-4}$ [10]
3	—	$(5.341 \pm 0.043) \times 10^{-4}$ [10]
4	$(5.690 \pm 0.027) \times 10^{-4}$ [8]	$(5.736 \pm 0.046) \times 10^{-4}$ [10]
∞	$(1 + 2\ln 2)/[18(4\pi)^2] \approx 8.39521 \times 10^{-4}$ [10]	

MC [4]: $\bar{c}_1 = 1.29 \pm 0.05$
MC [5]: $\bar{c}_1 = 1.32 \pm 0.02$
VPT (7 loops) [9]: $\bar{c}_1 = 1.27 \pm 0.11$
many other results: $\bar{c}_1 = -0.93, \ldots, 4.7$

MC [7]: $\bar{c}_2'' = 75.7 \pm 0.4$
VPT (7 loops) [10]: $\bar{c}_2'' = 74.6 \pm 2.3$
modified LDE [11]: $\bar{c}_2'' \approx 70, \ldots, 80$

MC [6]: $\breve{c}_2'' = -155.0 \pm 0.1$
VPT (7 loops) [10]: $\breve{c}_2'' = -155.0 \pm 0.7$

example: $\bar{c}_1$ from VPT in 4,5,6,7 loops
(lower/upper lines: fixed known $\omega = \frac{2}{2-\eta}\beta'(g^*)$ / self-consist. ω)



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