Quantum theory of a vortex line in an optical lattice

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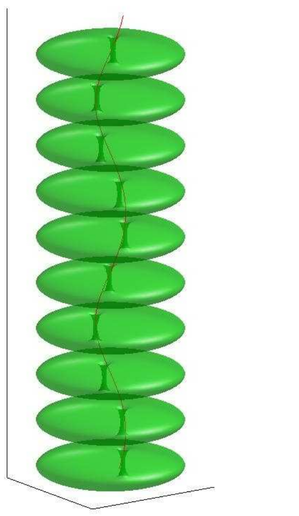
1 Introduction

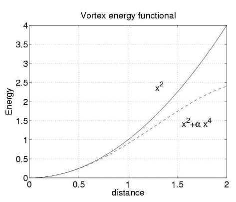
We study a vortex line in a one-dimensional optical lattice. Optical lattice splits the Bose-Einstein condensate into a stack of weakly coupled pancake shaped condensates, each being pierced by the vortex at some position (x_n, y_n). We assume that the lattice potential is deep enough so that the condensate wave function is frozen in each site in the axial direction, but still so shallow that we are in the superfluid regime instead of the Mott-insulator state.

Our aim is to discuss the quantum properties of the vortex line. Our approach naturally gives the dispersion relation for the vortex line eigenmodes (Kelvin modes [1, 2, 3]), but it also enables us to go further and discuss possible non-classical states of the vortex line as well as simple soliton solutions to the complicated problem of vortex line dynamics. Kelvin modes have been recently observed experimentally in a cigar shaped condensate without an optical lattice [2].



2 Vortex energy functional

Using a variational wave function in each site we calculate the energy of the vortex line as a function of on-site displacement from the condensate center as well as the strength of the inter layer coupling. The qualitative behavior of the energy functional is demonstrated in the figure.



To the leading order the on-site energy functional is harmonic. However, there is also a (small) quartic term which can be either positive or negative depending (among other things) on the rotation frequency of the trap. This term is always negative when the vortex is locally energetically stable.

3 Kelvin mode dispersion

It is important to note that vortex x and y coordinates are canonically conjugate and obey the uncertainty relation

$$[\hat{x}_n, \hat{y}_n] = iR^2/2N.$$

By defining creation and annihilation operators in terms of the vortex positions [1] as $\hat{x}_n = R/2\sqrt{N}\left(\hat{v}_n^\dagger + \hat{v}_n\right)$ and $\hat{y}_n = iR/2\sqrt{N}\left(\hat{v}_n^\dagger - \hat{v}_n\right)$ (R is the radial size of the condensate wave function and N is the number of atoms in each site) the vortex Hamiltonian takes the simple form [4, 5]

$$H_0 = \sum_n \left[\hbar\omega_0 + J_V\right] \hat{v}_n^\dagger \hat{v}_n - \frac{J_V}{2} \sum_{\langle n,m\rangle} \hat{v}_m^\dagger \hat{v}_n,$$

where ω_0 is the precession frequency of the straight vortex line and J_V is the strength of the coupling between nearest neighbor vortices. This can be easily diagonalized and we get a Kelvin mode dispersion

$$\hbar\omega_K(k) = \hbar\omega_0 + J_V\left(1 - \cos kd\right),$$

where d is the lattice spacing. Using a wave function ansatz with a Gaussian density distribution, we can calculate analytically the quantities ω_0 and J_V and relate them to system parameters such as the scattering length, trap frequencies, and the parameters of the optical lattice.

Note! Experimentally kelvons were observed by studying their coupling to the quadrupole modes (with $m = -2$ in particular). Similar setup can be conveniently studied using our variational approach [4, 5]. This gives an interaction Hamiltonian of type $\left(\hat{q}_{-2}\hat{v}_k^\dagger\hat{v}_{-k}^\dagger + h.c.\right)$ This is essentially the squeezing Hamiltonian of quantum optics.

4 Bose-Hubbard model

The Kelvin modes were extracted from the theory which is of second order in the vortex displacements. However, higher order terms do exist and the next non-vanishing term is of fourth order in the displacements. Expressing this term in terms of the kelvon operators reveals that up to this order the physics of the vortex line is described by the 1D Bose-Hubbard model. The relevant Hamiltonian is [6, 7]

$$\hat{H} = \hat{H}_0 + \frac{V_0}{2} \sum_n \hat{v}_n^\dagger \hat{v}_n^\dagger \hat{v}_n \hat{v}_n,$$

where V_0 is the interaction strength. This strength is always negative, when the vortex is energetically stable. However, for slowly rotating condensates it can be positive.

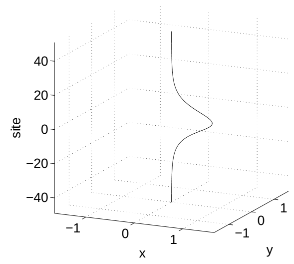
5 Vortex line solitons

At classical level one replaces all the operators $\hat{v}_n$ with complex numbers. In that case the equations of motion is the discretized version of the 1D Gross-Pitaevskii equation. When gradients are small and interaction is attractive there is the well known bright soliton solution [7]

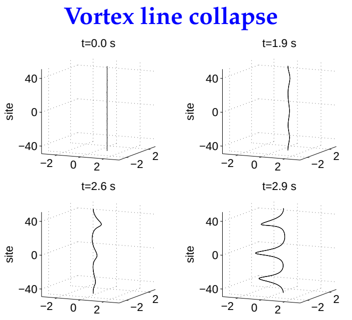
$$v_n(t) = \sqrt{\frac{N_0}{2\xi}} \frac{e^{-i\mu t + ikn}}{\cosh\left[(n - J_V k t/2)/\xi\right]},$$

where k is the wavevector and the size of the soliton is $\xi = 2J_V/|V_0|N$ lattice spacings. This soliton propagates without changing its shape. When the soliton does not propage in the lattice direction, it corresponds to a curved vortex line precessing around the condensate, with a frequency that is somewhat smaller than the precession frequency of the straight vortex line. Initially straight but displaced vortex line can be dynamically unstable and collapse into a train of bright solitons. Such a process is demonstrated in the figure below.

Bright vortex line soliton



Vortex line collapse



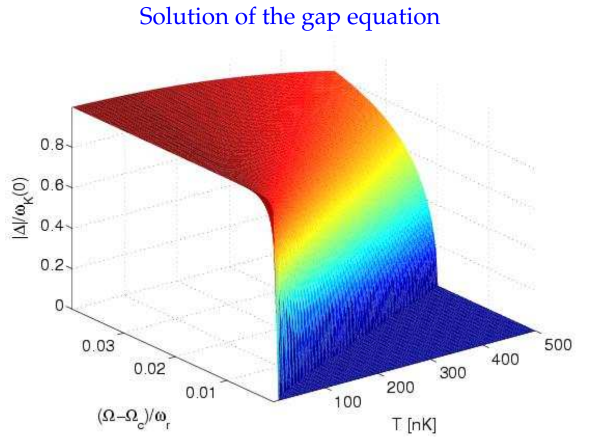
6 Squeezed vortex states

Vortex line solitons are excited states of the vortex line. We now turn to the equilibrium properties. Since we assume energetically stable vortices $\langle\hat{v}_n\rangle = 0$, since otherwise we are dealing with a displaced vortex line. However, we can identify the order parameter with $\Delta = V_0\langle\hat{v}_n\hat{v}_n\rangle$ [6]. Expanding the Hamiltonian quadratically around this order parameter, we get a Hamiltonian which can be diagonalized by means of a Bogoliubov transformation, analogous to the BCS theory. Furthermore, self-consistency requirement leads to a gap equation.

$$\frac{1}{V_0} = -\frac{1}{N_s}\sum_k \frac{1 + 2N_k}{2E(k)},,$$

where $N_k = 1/(e^{\beta E(k)} - 1)$ is the Bose distribution and $E(k) = \hbar\sqrt{\omega_K(k)^2 - |\Delta|^2}$ is the dispersion of the Bogoliubov quasiparticles. We then solve the gap equation for the order parameter.

Solution of the gap equation



The result is given in the space spanned by the rotation frequency Ω and the temperature T. Ω_c is the rotation frequency above which the vortex is locally energetically stable. The order parameter is bounded from above by the precession frequency ω_0 and therefore the result is plotted in terms of $\Delta/\hbar\omega_0$.

7 Meaning of squeezing

The complex order parameter $\Delta = |\Delta|e^{i\phi}$ has an interesting physical interpretation in terms of the quantum mechanical uncertainty of the vortex position. It turns out that in a coordinate system rotated by an angle θ we have

$$\langle\hat{y}_n^2\rangle - \langle\hat{x}_n^2\rangle = |\Delta| \times \left(\frac{R^2\cos(\phi - 2\theta)}{|V_0|NN_s}\right),$$

where N_s is the number of lattice sites. Therefore, squeezing will be reflected in the squeezing of the vortex position distribution.

When the phase fluctuations are negligible, the uncertainty ellipse of the vortex position distribution is independent of the layer index n. Therefore, the measurement of the vortex positions in different layers samples the same distribution and provides a signature for the expected transition into a squeezed state. Position distribution can become so strongly squeezed that it should be observable. Also, the radial condensate expansion will only change the vortex distribution by a scale factor and will not wash out the effect.

8 Phase fluctuations

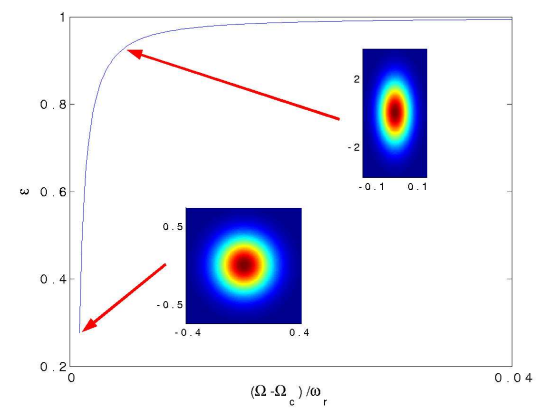
In an infinite one-dimensional system phase fluctuations destroy the long-range order. In the finite system we are considering here, phase fluctuations can only be excited if the temperature is high enough In order to calculate the energy cost for a phase gradient of the order parameter we must determine the associated stiffness or superfluid density $\rho_S(T)$. Since the lattice breaks the Galilean invariance this calculation is not entirely standard. However, after some work the result can be expressed in terms of the Bogoliubov amplitudes [8, 6].

$$\rho_s(T) = \frac{1}{2N_s d}\sum_k \left\{\cos(kd)\left[|u_k|^2 N_k + |v_k|^2(N_k + 1)\right] - J_V\beta\sin^2(kd)\,N_k(N_k + 1)\right\}.$$

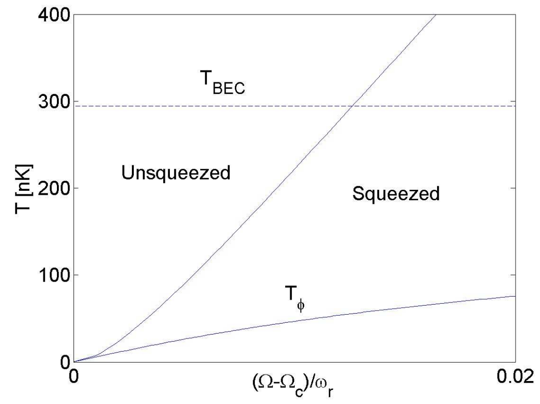
In an appropriate limit this expression reproduces the well known Landau result for the superfluid density. Equating the energy cost due to a $2\pi/N_s d$ phase gradient with the thermal energy gives us an estimate for the temperature scale $T_\phi = J_V\rho_S(T)\pi^2 d/N_s k_B$ of the phase fluctuations. Our results are summarized in the following phase diagram.

Relative and absolute squeezing of the coordinates

$$\epsilon = \frac{\langle x^2\rangle - \langle y^2\rangle}{\langle x^2\rangle + \langle y^2\rangle}$$



Phasediagram



7 Conclusions

- Bose-Hubbard model for the vortex line
- Soliton solutions in the classical limit
- Squeezing possible in equilibrium
- Future: vortex lattices, new phase transitions, new type of solitons due to the vortex-vortex interactions?

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