Quantum Simulator for the Ising Model with Electrons Floating on a Helium Film

Sarah Mostame\(^1\) and Ralf Schützhold\(^1,2,\ast\)

\(^1\)Institut für Theoretische Physik, Technische Universität Dresden, 01062 Dresden, Germany
\(^2\)Fachbereich Physik, Universität Duisburg-Essen, 47048 Duisburg, Germany

(Received 16 June 2008; published 26 November 2008)

We propose a physical setup that can be used to simulate the quantum dynamics of the Ising model in a transverse field. Building on currently available technology, our scheme consists of electrons which float on a superfluid helium film covering a suitable substrate and interact via Coulomb forces. At low temperatures, the system will stay near its ground state where its Hamiltonian is equivalent to the Ising model and thus shows phenomena such as quantum criticality.

\textbf{Introduction.}—Richard Feynman’s observation [1] that classical computers cannot effectively simulate quantum systems bred widespread interest in quantum computation. He thought up the idea of a quantum processor which uses the effects of quantum theory instead of classical physics. As an example, Feynman proposed a universal quantum simulator consisting of a lattice of spins with nearest-neighbor interactions that are freely specifiable and can efficiently reproduce the dynamics of any other many-particle quantum system with a finite-dimensional state space [1]. Although such universal quantum computers of sufficient size (e.g., number of qubits, i.e., spins) are not available yet, it is possible to design a special quantum system in the laboratory which simulates the quantum dynamics of a particular model of interest. Such a designed quantum system can then be regarded as a special quantum computer (instead of a universal one, which is more challenging) which just performs the desired quantum simulation, see, e.g., [2–4].

In the following, we present a design for a quantum simulator for the Ising spin chain in a transverse field and demonstrate that it could be feasible with near-future technology, i.e., electrons floating on a thin superfluid helium film. A similar idea based on trapped ions has been pursued in [2]. Nevertheless, it is still worthwhile to study an alternative set-up, since different experimental realizations possess distinct advantages and drawbacks. For example, the number of coherently controlled ions in a trap is rather limited at present [2], whereas our proposal is expected to be more easily scalable to a large number of electrons—which is important for exploring the continuum limit and scaling properties, etc.

\textbf{The model.}—Our aim is to simulate the quantum dynamics of the one-dimensional Ising chain consisting of \(n\) spins with nearest-neighbor interaction \(J\) plus a transverse field \(\Gamma\) along the \(x\) direction (\(h = 1\))

\[ H = -\sum_{j=1}^{n} \left( \Gamma \sigma_j^x + J \sigma_j^z \sigma_{j+1}^z \right), \tag{1} \]

where \(\sigma_j = (\sigma_j^x, \sigma_j^y, \sigma_j^z)\) are the spin-1/2 Pauli matrices acting on the \(j\)th qubit. This model has been employed in the study of quantum phase transitions and percolation theory [5], spin glasses [5,6], as well as quantum annealing [7,8] etc. Although the Hamiltonian (1) is quite simple and can be diagonalized analytically, the Ising model is considered a paradigmatic example [5] for second-order quantum phase transitions and is rich enough to display most of the basic phenomena near quantum critical points. For \(\Gamma \gg J\), the ground state is paramagnetic \(\left\langle \sigma_j^z \right\rangle \rightarrow 0\) with all spins polarized along the \(x\) axis. In the opposite limit \(\Gamma \ll J\), the nature of the ground state(s) changes qualitatively and there are two degenerate ferromagnetic phases with all spins pointing either up or down along the \(z\) axis \(\left\langle \sigma_j^z \right\rangle \rightarrow \pm 1\) or \(\left\langle \sigma_j^z \right\rangle \rightarrow 0\) respectively. The two regimes are separated by a quantum phase transition at the critical point \(\Gamma_c = J\), where the excitation gap vanishes (in the thermodynamic limit \(n \rightarrow \infty\)) and the response time diverges. As a result, driving the system through its quantum critical point at a finite sweep rate entails interesting nonequilibrium phenomena such as the creation of topological defects, i.e., kinks [9]. Furthermore, the transverse Ising model can also be used to study the order-disorder transitions at zero temperature driven by quantum fluctuations [5,7]. Finally, two-dimensional generalizations of the Ising model can be mapped onto certain adiabatic quantum algorithms (see, e.g., [10]). However, due to the evanescent excitation energies, such a phase transition is rather vulnerable to decoherence, which must be taken into account [11].

\textbf{The analogue.}—In order to reproduce the quantum dynamics of the \((1 + 1)\)-dimensional Ising model (1), we propose trapping a large number of electrons on a low-temperature helium film of thickness \(h\) (e.g., \(h = 110\) nm) adsorbed on a silicon substrate [12]. Because of the polarizability \(\varepsilon \approx 1.06\) of the helium film, the electrons are bound to its surface (i.e., in \(z\) direction) via their image charges and the large potential barrier (around 1 eV) for penetration into the helium film [13]. Since the binding energy of around 8 K is much larger than the temperature \(T\) (below 1 K) and the width of the electron wave packet in \(z\) direction (of order 8 nm) is much smaller than all other relevant length scales, the electron motion is approximately two-dimensional \((x, y\) plane).
In our scheme, each single electron on top of the helium film is trapped by a pair of (grounded) gold spheres of radius $a$ (e.g., $a = 10$ nm) and distance $d$ (e.g., $d = 60$ nm) attached to the silicon substrate (i.e., on the bottom of the helium film cf. Fig. 1). Depending on its position $x$, $y$, the electron will also induce image charges in the two gold spheres (which act as a pair of quantum dots) and hence experience a double-well potential

$$U_w(x, y) = -\frac{ae^2(x^2 + y^2 + \alpha^2 + \beta^2)/4\pi\varepsilon}{(x^2 + y^2 + \alpha^2 + \beta^2)^{3/2} - 4\alpha^2y^2},$$

with $\alpha = d/2 + a$ and $\beta^2 = h^2 - a^2$. Since this potential is quite deep and symmetric $U(x, y) = U(x, -y)$ cf. Fig. 1, the ground-state wave function $\psi_+(x, y)$ is given by the symmetric superposition of the two Wannier states $\psi_0(x, \pm y)$ while the first excited state $\psi_-(x, y)$ is the antisymmetric combination

$$\psi_{\pm}(x, y) = \frac{\psi_0(x, y) \pm \psi_0(x, -y)}{\sqrt{2}} \mp \frac{|1\rangle \pm |1\rangle}{\sqrt{2}}.$$

For a sufficiently high potential barrier between the two wells, the Wannier state $\psi_0(x, y)$ is strongly concentrated in the left well and models the spin state $|1\rangle$ and vice versa. The tunneling between the two states is then described by the Pauli operator $\sigma^z$ with $\sigma^z|1\rangle = |1\rangle$ and $\sigma^z|\downarrow\rangle = \mp |\downarrow\rangle$ such that the tunneling rate, given by the difference of the eigenenergies $E_A - E_S$ of $\psi_+$ and $\psi_-$, corresponds to the transverse field $\Gamma$ in Eq. (1). In the limit of strong localization (i.e., weak tunneling), the energy splitting $E_A - E_S$ between the two levels can be estimated via the WKB approximation

$$2\Gamma = E_A - E_S = \frac{\omega}{\pi} \exp\left[-\int_{-y_0}^{y_0} dy |p(y)| \right].$$

Here $\omega$ is the oscillation frequency (within one well) and $\pm y_0$ are the two inner (classical) turning points cf. Fig. 1. The integrand is given by $p(x, y) = \sqrt{2m_e(E_0 - U(x, y))}$, where we can set $x = 0$ since the tunneling probability away from the $x = 0$ axis is strongly suppressed. Finally, the energy $E_0$ determines the turning points and $m_e$ is the electron mass. For the parameters above, each valley can be well approximated by a harmonic oscillator $U_w(x, y = \pm y_{\text{min}}) = ae^2(x^2 + [(y + y_{\text{min}})]^2)/(4\pi\varepsilon\beta^4)$, and thus we obtain $E_0 \approx \sqrt{ae^2/2\pi\varepsilon m_e\beta^4} = \omega$.

So far, we derived the term $\Gamma\sigma^z_1$ in Eq. (1) via Eqs. (2)–(4). In order to simulate the remaining part, we propose to line up the pairs of quantum dots at equal distances $\lambda$ (e.g., $\lambda = 600$ nm), where the parameters are supposed to obey the following hierarchy

$$\lambda \gg h > d \gg a.$$

Within this limit, the interaction between the electrons will be dominated by the direct Coulomb repulsion between the nearest neighbors $U_j = \sum_{j=1}^n U_{j;j+1}^{n}$ with $n$ denoting the number of electrons floating on the helium film. For $\lambda > d$, we may Taylor expand the Coulomb interaction into powers of $y/\lambda$ due to $y = \pm d/2$. The zeroth-order term is constant and thus irrelevant while the first-order contributions vanish (up to boundary terms) after the sum over sites $j$. Thus, the leading term is bilinear in the electron positions

$$U_c(x, y) = -\frac{e^2}{2\pi\varepsilon_0(\lambda + d + 4a)^3} \sum_{j=1}^n y_j y_{j+1},$$

FIG. 1 (color online). Sketch of the proposed analogue quantum simulator and the induced double-well potential $U(y)$ with four turning points for the energy $E_0$. Electrons ($e^*$) are floating on a low-temperature helium film of height $h$ adsorbed on a silicon substrate. A double-well potential for each single electron is created by a pair of (grounded) gold spheres of radius $a$ and distance $d$ on the bottom of the helium film. The double wells at each site provide the two lowest states of the electron and model the spin state $|\uparrow\rangle$ and $|\downarrow\rangle$ at each site $j$. The tunneling rate between the two wells corresponds to the transverse field $\Gamma\sigma^z_j$. The electrons are lined up at distances $\lambda$ and interact via Coulomb forces, which creates the term $J\sigma^z_j\sigma^z_{j+1}$. Alternative scenarios are sketched in Fig. 2.

FIG. 2 (color online). Sketch of various arrangements: (a) the line setup of Fig. 1 and (b) the alternative ladder setup discussed after Eq. (6), both for the one-dimensional Ising model (1), as well as their two-dimensional combination (c). Using the same distances $\lambda = 600$ nm and $\lambda' = 560$ nm as in (a) and (b), setup (c) corresponds to the symmetric ferromagnetic interaction $J\sigma^z_j\sigma^z_{j+1}$ to lowest order—if we use the same sign for $\sigma^z_j$ within one (horizontal) line and alternating signs in neighboring lines. After a checkerboard sign redefinition, the nearest-neighbor interaction becomes antiferromagnetic. Varying the ratio $\lambda'/\lambda$, we may induce an asymmetry $J_\uparrow \neq J_\downarrow$. The tunneling term $\Gamma\sigma^z_j$ is the same in all cases.
and precisely corresponds to the $J\sigma_j^z\sigma_{j+1}^z$ term in Eq. (1) with $J = e^2/(d + 2a)^2/(8\pi\varepsilon_0[a + d + 4a^3])$. As an alternative setup, one may arrange the pairs of quantum dots in parallel (i.e., as a ladder) cf. Fig. 2. Adjusting the distance accordingly (e.g., $\lambda' = 560$ nm instead of $\lambda = 600$ nm), we obtain the same coupling strength $J$ at lowest order, provided that $\sigma_j^z$ is identified with the electron position in alternating order. Combining the line and the ladder design then facilitates the realization of the two-dimensional Ising model; cf. Fig. 2.

**Experimental parameters.**—For the example values given in the text, we obtain $\Gamma \approx 0.1$ K for the tunneling rate and the same value $J \approx 0.1$ K for the effective coupling; i.e., we are precisely in the quantum critical regime. However, deviations from this critical point should be easy to realize experimentally by varying the height $h$ of the helium film, e.g., via changing the effective chemical potential difference between the thin helium film and the helium reservoir (which can be done by depleting the reservoir, for example). This is possible because the tunneling rate depends strongly—in fact, exponentially—on $h$ (e.g., varying $h$ by 10%, $\Gamma$ changes by 50%), whereas the Coulomb force $J$ remains approximately constant. In order to see quantum critical behavior, i.e., to avoid thermal fluctuations, the temperature should ideally be well below this value 0.1 K (or at least not far above it).

Furthermore, the Coulomb repulsion energy between two electrons (zeroth-order term) of about 11 K (for $\lambda = 600$ nm) would tend to destabilize the electron chain. Fortunately, this effect is compensated by the binding energy between the electron and its image on the sphere, which is around 13 K and thus stabilizes the electron chain. Note that the image charge on the sphere is much smaller than 1 $e$ due to $h \gg a$. The probability for the electron to penetrate the helium film by tunneling to one of the gold spheres is extremely small (of order $10^{-16}$) and can be neglected. Finally, the ground-state energy $E_0 \approx 1.4$ K (within the harmonic oscillator approximation) is reasonably well below the barrier height $U_0 \approx 3.1$ K such that the WKB approximation should provide a reasonable estimate. (The tunneling probability of 0.08 is also small enough.) On the other hand, $E_0 \approx 1.4$ K is a measure of the distance between the two lowest-lying states in Eq. (3) and the remaining excited states in the double-well potential. As a result, these additional states do not play a role for temperatures well below 1 K and thus the Hamiltonian (1) provides the correct low-temperature description.

**Read-out scheme.**—Having successfully simulated the Ising Hamiltonian (1), one is led to the question of how to actually measure its properties, e.g., how to detect signatures of quantum critical behavior. As one possibility, let us imagine that the gold spheres are not grounded, but connected to small wires which allow us to address them individually or in suitable partitions. For example, applying a voltage of one $\mu$V between the spheres associated with spin up ($\uparrow$) and spin down ($\downarrow$), respectively, induces a perturbation Hamiltonian $\gamma\sigma_j^z$ corresponding to a longitudinal field (in addition to the transversal one $\Gamma\sigma_j^z$) of $\gamma = O(10^{-3}$ K), i.e., a weak perturbation $\gamma \ll \Gamma$. Deep in the paramagnetic phase $\Gamma \gg J$, the response of the system to this weak perturbation $\gamma \ll \Gamma$ is rather small ($\langle \sigma_j^z \rangle \approx \gamma/\Gamma$). Approaching the phase transition, however, the static susceptibility $\chi_\gamma = \lim_{\gamma \to 0}(\langle \sigma_j^z \rangle)/\gamma$ grows and finally diverges at the critical point. In the broken symmetry phase, the perturbation $\gamma\sigma_j^z$ lifts the degeneracy $\sigma_j^z \rightarrow -\sigma_j^z$ and hence the response is nonanalytic, i.e., independent of the smallness of $\gamma$: e.g., for $J \gg \Gamma$, we have $\langle \sigma_j^z \rangle = \text{sgn}(\gamma) = \pm 1$.

This signal $\langle \sigma_j^z \rangle$ indicating the phase transition can be picked up by measuring the voltage difference induced by the position of the electron. If the spheres are not grounded but isolated, the electron induces a voltage difference of up to 13 mV between the two spheres. Clearly, this would constitute a large disturbance and thus one should put a resistor or capacitor between them in order to reduce the voltage difference to a few $\mu$V. However, since the associated charge transfer is small (a fraction of the elementary charge $e$), a site-by-site readout is probably very hard to realize. Nevertheless, collecting signals from many electrons should lead to measurable currents. Preferably, this measurement should be done using other spheres than the one on which the external voltage has been applied. E.g., one could apply the external voltage of one $\mu$V on half of the electrons and measure (with suitably adapted internal resistances) the internal response $\langle \sigma_j^z \rangle$ on the other half—or in large enough subsets.

Comparing the signals from the different partitions then yields information about the correlator $\langle \sigma_i^z\sigma_j^z \rangle$. In addition to the static case, one could also study the time-resolved response $\langle \sigma_j^z(t) \rangle$ to a varying voltage $\gamma(t)$, which is determined by the dynamical correlator $\langle \sigma_i^z(t')\sigma_j^z(i) \rangle$ in lowest-order response theory. Even in the absence of an externally imposed voltage, the chain induces spontaneous voltage fluctuations in the electrodes, which are strongest deep in the ferromagnetic phase. The variance of these fluctuations again yields information about the correlator $\langle \sigma_i^z\sigma_j^z \rangle$ which is an order parameter for the phase transition and allows us to detect topological defects (i.e., kinks) which might have been produced during the sweep to the ferromagnetic phase.

**Disorder and decoherence.**—In a real experimental setup, the Hamiltonian will not be exactly equivalent to (1) due to imperfections such as electric stray fields, variations in the film thickness $h$ and further geometric parameters $a, d,$ and $\lambda$ etc. Therefore, the original expression (1) will typically be altered to $H = -\sum\{\Gamma_j\sigma_j^z + J_j\sigma_j^z\sigma_{j+1}^z + \gamma_j\sigma_j^z\}$, where $\Gamma_j = \bar{\Gamma} + \delta\Gamma_j$ and $J_j = \bar{J} + \delta J_j$. Assuming that the disorder parameters $\delta\Gamma_j,$ $\delta J_j,$ and $\gamma_j$ are much smaller than the excitation gap $\Delta = 2|\bar{J} - \bar{\Gamma}|$ of the undisturbed system (in the continuum limit), the impact of these imperfections will be sup-
pressed. Near the critical point $J = \Gamma$; however, this argument fails. Nevertheless, for a finite number of electrons, one retains a minimum gap (within the symmetric or anti-symmetric subspace, respectively) of order $J/n$. Exploiting this gap might be suitable for a reasonably small systems, but for $n \approx 100$ electrons, the required accuracy on the sub-percent level is probably hard to achieve experimentally. E.g., decreasing the diameter of the gold spheres by 10% with the other values remaining the same, the tunneling rate increases by 50%.

For a sufficiently large number of electrons, the relevance of the disorder induced by imperfections near the critical point depends on the dimensionality of the system. In one spatial dimension, the critical exponent $\nu = 1$ of the Ising model indicates that the renormalization flow is directed away from the homogeneous situation, see, e.g., [5], which means that disorder becomes important at large scales $n \rightarrow \infty$. In this case, one would expect effects such as local paramagnetic regions inside the global ferromagnetic phase and percolation transitions, etc. Therefore, turning this drawback into an advantage, one might generate these imperfections on purpose in order to study the impact of disorder on the phase transition. In contrast to the original Hamiltonian (1), the above form in the presence of disorder is no longer analytically solvable and hence much less is known about its properties. In two spatial dimensions (cf. Fig. 2), the relevance of disorder is much weaker and hence it might be possible to observe quantum critical behavior.

Finally, in a real setup, the system will also experience decoherence due to the inevitable coupling to the environment [11]. These effects could be incorporated by operator-valued variations $\delta \Gamma_j$, $\delta J_j$, and $\gamma_j$ associated to the degrees of freedom of the environment. According to [15], the main decoherence channels in the setup under consideration are due to the coupling to ripplons, i.e., surface waves on the helium film. At $T = 0.1$ K, their thermal fluctuations have a typical amplitude of $\delta h = \sqrt{k_B T/\xi} \approx 0.06$ nm, where $\xi$ is the surface tension. Since $\delta h$ is extremely small compared to $h = 110$ nm, the coupling terms and energy shifts induced by these height variations are negligible here. Furthermore, the energy $\omega$ of the ripplons itself depends on their wave number $k$ via $\omega^2 = k^3 \xi / \phi$, where $\phi$ is the density of the helium film. For typical wavelengths of the ripplons coupled to the electrons of order 100 nm, the ripplon energy is below 0.01 K and hence also negligible. Consequently, even though the ripplons might induce significant dephasing and decay of excited states ($T_2$ and $T_1$ cf. [15]), they basically do not affect the quantum ground state.

**Summary:**—We have proposed a design for the simulation of the quantum Ising model with a system of electrons floating on a liquid helium film adsorbed on a silicon substrate. Since the energy level splitting (tunneling rate $\Gamma$) depends exponentially on the thickness of the helium film $h$, we may tune the system through the quantum phase transition by changing $h$—which might even be feasible in a time-dependent manner, cf. [9]. The quantum critical behavior and the created topological defects (kinks) could be detected via measuring the voltages induced on the spheres. Furthermore, a suitable generalization to two spatial dimensions might be relevant for adiabatic quantum algorithms, see, e.g., [10]. Note that the realization of a sequential quantum computer based on a set of electrons floating on a helium film has been proposed in [15]. In contrast, our proposal is not suited for universal computations, but (as one would expect) should be easier to realize experimentally. Exploring a different limit, where many eigenstates of the double-well potential contribute, the proposed set-up could simulate the lattice version of interacting field theories such as the $\lambda \phi^4$-model in $(1 + 1)$ dimensions.

S. M. acknowledges fruitful discussions with R. Farhadifar and R. S. is indebted to G. Volovik and P. Leiderer for valuable conversations. This work was supported by the Emmy-Noether Programme of the German Research Foundation (DFG) under Grant SCHU 1557/1-2,3 and by DFG Grant SCHU 1557/2-1.

*ralf.schuetzhold@uni-due.de*