Complex classical mechanics

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Dresden 2011

PT quantum mechanics has an active research community

Nearly1000 published papers!

Lots of Conferences!
Andrianov, Caliceti, Fring, Gazeau, Geyer, Jain, Jones,
Mostafazadeh, Rotter, Scholz, Wu, Znojil, ….

Webpage!
Hook: PT Symmeter http://ptsymmetry.net

Google “PT Symmetric” --- 72,500 hits!

recent *PT* papers …

- K. Makris, R. El-Ganainy, D. Christodoulides, and Z. Musslimani, *Phyical Review Letters* **100**, 103904 (2008)
- Z. Musslimani, K. Makris, R. El-Ganainy, and D. Christodoulides, *Physical Review Letters* **100**, 030402 (2008)
- U. Günther and B. Samsonov, *Physical Review Letters* **101**, 230404 (2008)
- E. Graefe, H. Korsch, and A. Niederle, *Physical Review Letters* **101**, 150408 (2008)
- S. Klaiman, U. Günther, and N. Moiseyev, *Physical Review Letters* **101**, 080402 (2008)
- CMB and P. Mannheim, *Physical Review Letters* **100**, 110402 (2008)

- U. Jentschura, A. Surzhykov, and J. Zinn-Justin, *Physical Review Letters* **102**, 011601 (2009)
- A. Mostafazadeh, *Physical Review Letters* **102**, 220402 (2009)
- O. Bendix, R. Fleischmann, T. Kottos, and B. Shapiro, *Physical Review Letters* **103**, 030402 (2009)
- S. Longhi, *Physical Review Letters* **103**, 123601 (2009)
- A. Guo, G. J. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G. A. Siviloglou, and D. N. Christodoulides, *Physical Review Letters* **103**, 093902 (2009)

- H. Schomerus, *Physical Review Letters* **104**, 233601 (2010)
- S. Longhi, *Physical Review Letters* **105**, 013903 (2010)
- C. West, T. Kottos, T. Prosen, *Physical Review Letters* **104**, 054102 (2010)
- S. Longhi, *Physical Review Letters* **105**, 013903 (2010)
- T. Kottos, *Nature Physics* **6**, 166 (2010)
- C. Ruter, K. Makris, R. El-Ganainy, D. Christodoulides, M. Segev, and D. Kip, *Nature Physics* **6**, 192 (2010)
- CMB, D. Hook, P. Meisinger, Q. Wang, *Physical Review Letters* **104**, 061601 (2010)
- CMB and S. Klevansky, *Physical Review Letters* **105**, 031602 (2010)

- Y. D. Chong, L. Ge, and A. D. Stone, *Physical Review Letters* **106**, 093902 (2011)
- Z. Lin, H. Ramezani, T. Eichelkraut, T. Kottos, H. Cao, and D. N. Christodoulides, *Physical Review Letters* **106**, 213901 (2011)
- Another *Physical Review Letter* on the way by S. Rotter *et al*

PT quantum mechanics is fun because you can re-visit the things you already know about ordinary Hermitian quantum mechanics!

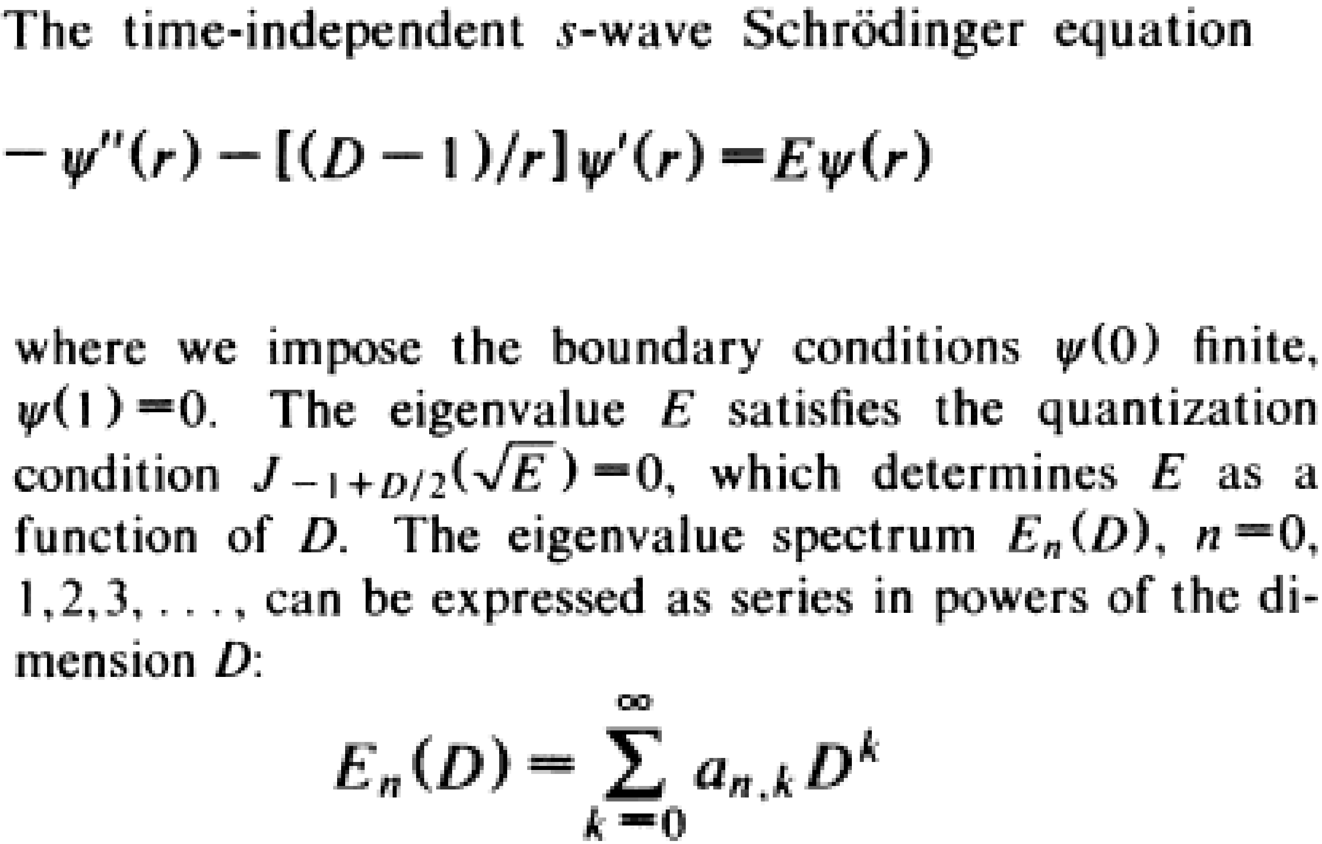
Here are some examples…

Example 1: Dimensional expansions

CMB, S. Boettcher, and L. Lipatov,
Physical Review Letters, **68**, 3674 (1992)

The idea:
Physics becomes simple near $D = 0$, so obtain a nonperturbative solution by expanding in powers of D

D-dimensional square well

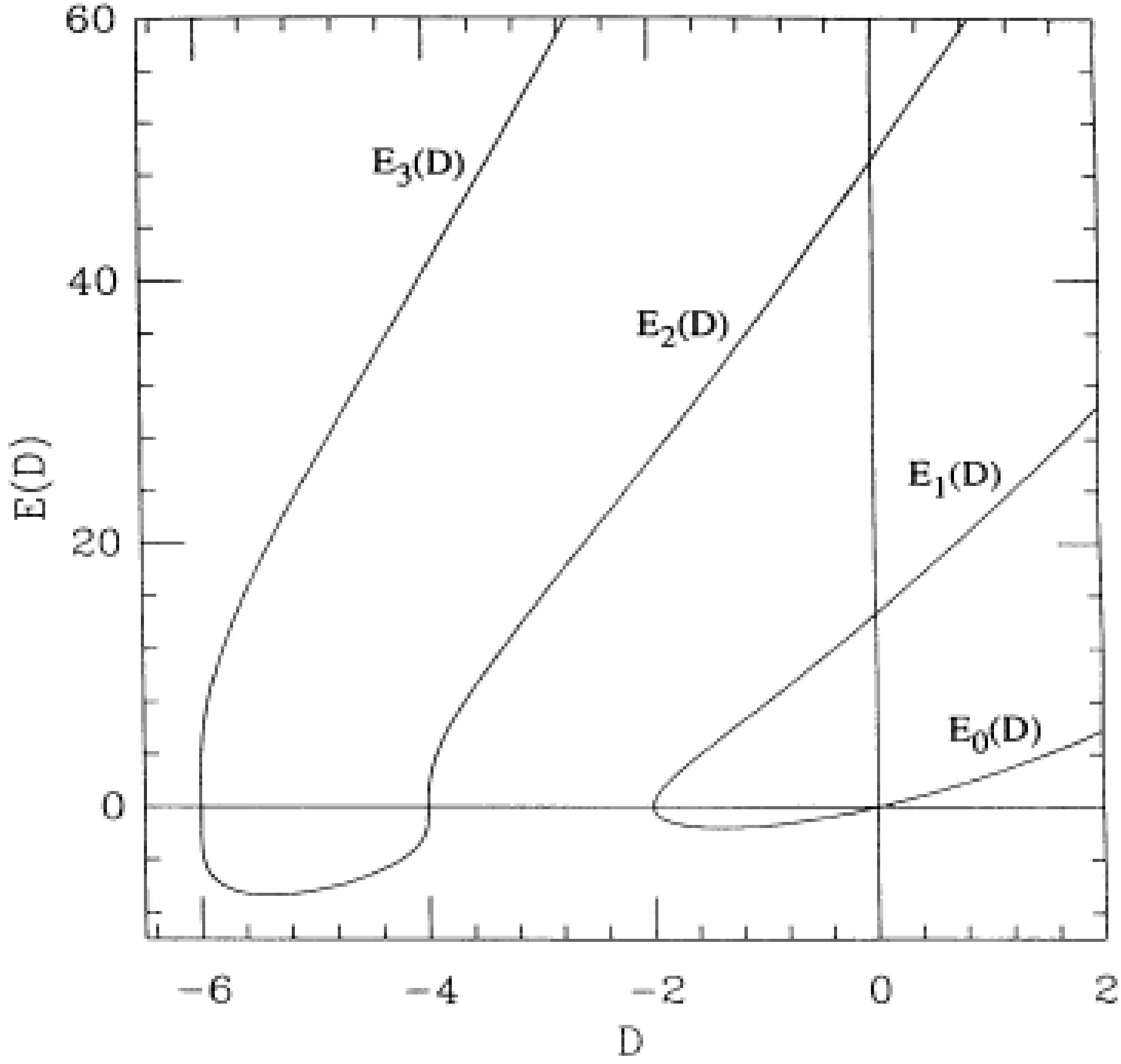


The time-independent *s*-wave Schrödinger equation

$$-\psi''(r)-[(D-1)/r]\psi'(r)=E\psi(r)$$

where we impose the boundary conditions $\psi(0)$ finite, $\psi(1)=0$. The eigenvalue E satisfies the quantization condition $J_{-1+D/2}(\sqrt{E})=0$, which determines E as a function of D. The eigenvalue spectrum $E_n(D)$, $n=0, 1,2,3,\ldots$, can be expressed as series in powers of the dimension D:

$$E_n(D)=\sum_{k=0}^{\infty} a_{n,k}D^k$$



Non-Hermitian Hermitian

Dimensional expansions in quantum field theory...

CMB, arXiv: hep-th/1003.3881

$$L = \tfrac{1}{2}(\partial\phi)^2 + g\phi^{2K}$$

Free energy (vacuum energy density) *F*: $e^{-FV} = \int D\phi \exp\left(-\int d^D x\, L\right)$

$$A(\epsilon, D) \equiv 2K g^{1-D/[2K-D(K-1)]} \frac{dF}{dg}$$

$$A_K(D) = \alpha + \beta D + \gamma D^2 + \cdots$$

$$A_K(D) = 1 - \frac{D}{2} \ln\left[8 e^{-\gamma} e^{-1+1/K} \Gamma^2 \left(1 + \frac{1}{2K}\right)\right] + \gamma D^2 + \cdots$$

PT quantum field theory

$$L=\tfrac{1}{2}(\partial\phi)^2+g\phi^2(i\phi)^\epsilon \qquad (\epsilon\geq 0)$$

$$e^{-FV}=\int_C \mathcal{D}\phi\,\exp\left(-\int d^Dx\,L\right)$$

$$A(\epsilon,D)\equiv(2+\epsilon)g^{1-D/[2+\epsilon-D\epsilon/2]}\frac{dF}{dg}.$$

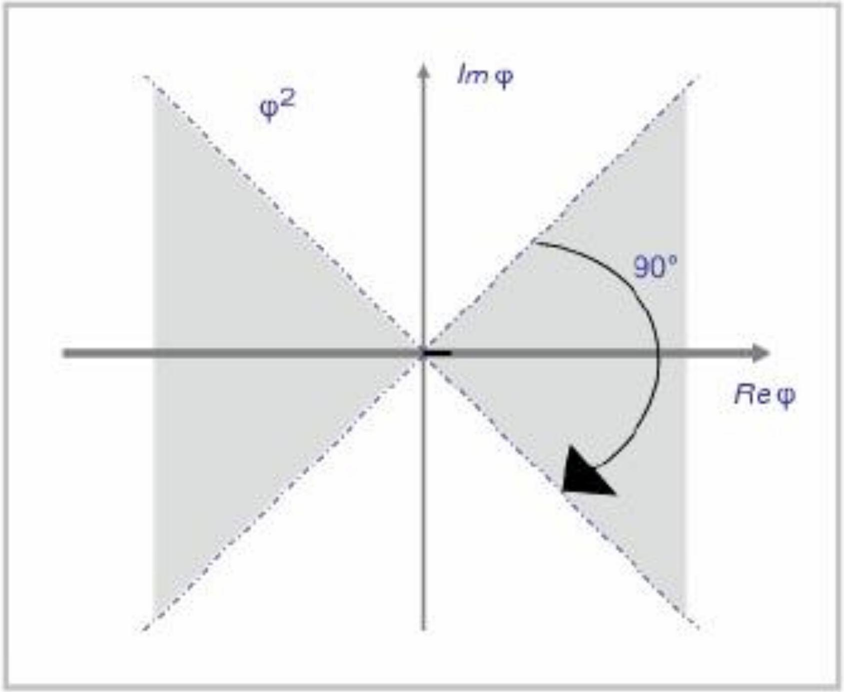
$$A(\epsilon,D)=1-\tfrac{1}{2}D\left[4e^{-\gamma}\Gamma^2\left(1+\tfrac{1}{2+\epsilon}\right)\cos^2\left(\tfrac{\pi\epsilon}{4+2\epsilon}\right)\right]+\mathrm{O}\left(D^2\right)$$

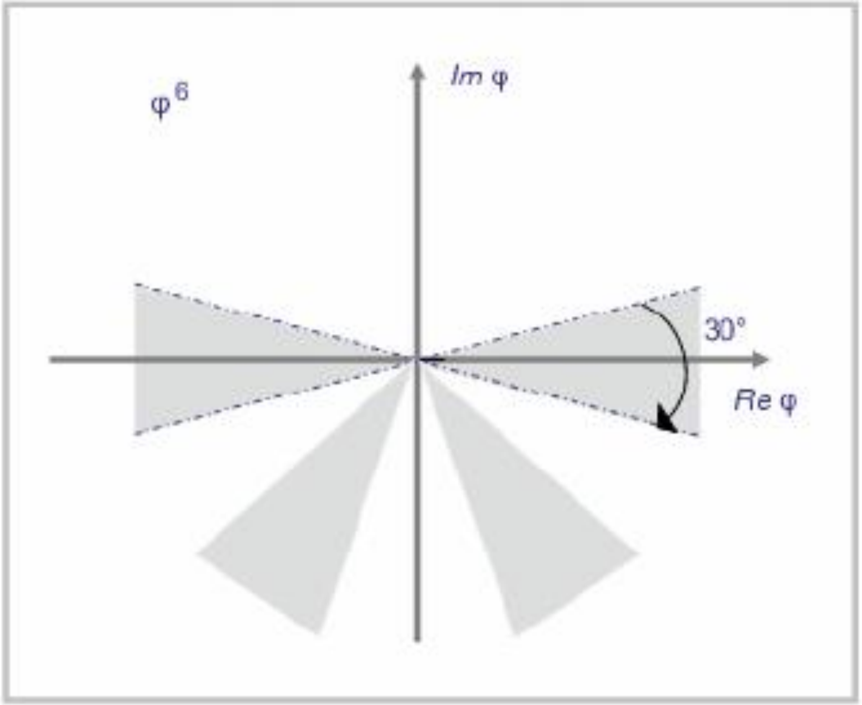
Example 2:Functional integrals

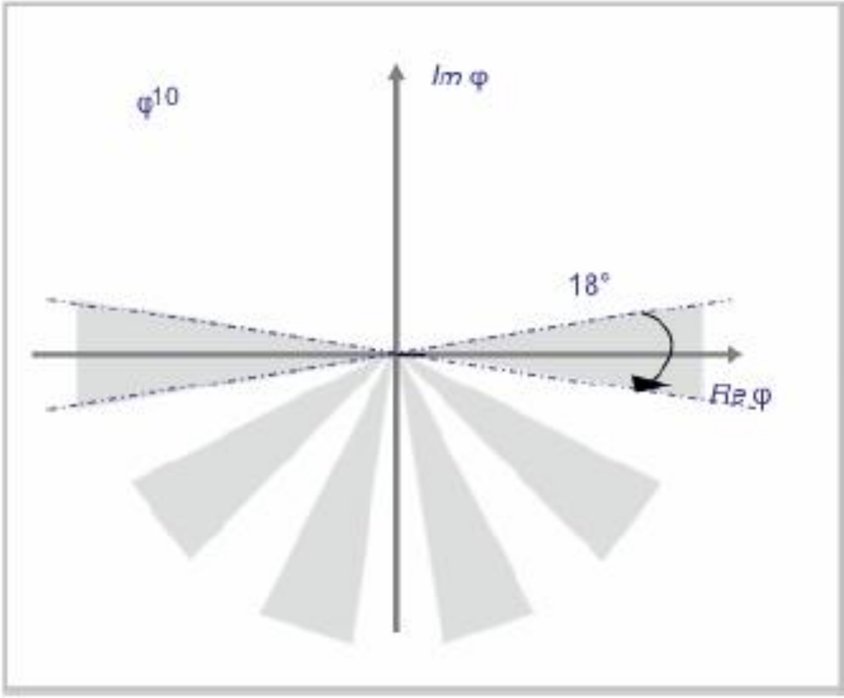
CMB and S. Klevansky,
Physical Review Letters **105**, 031602 (2010)

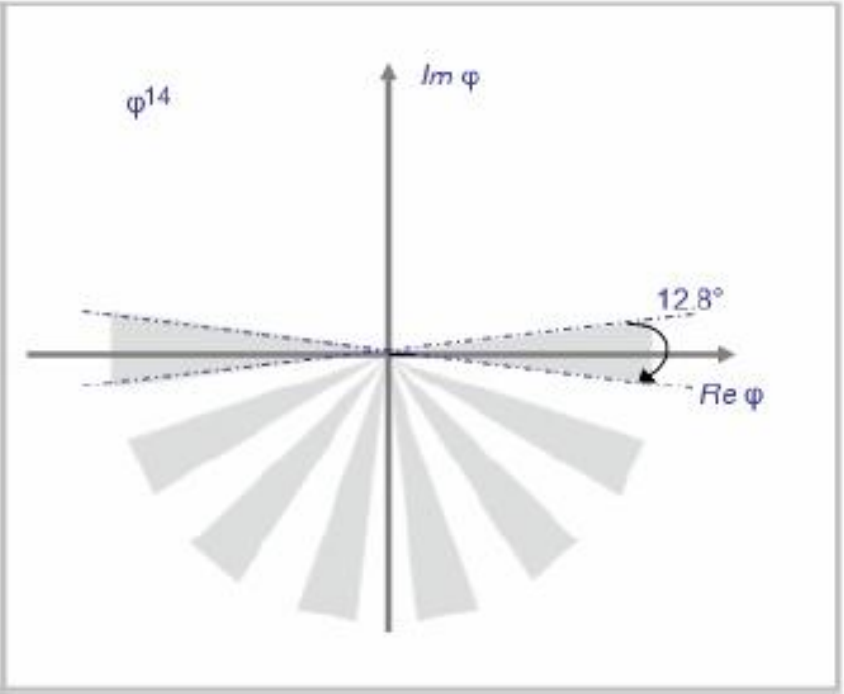
$$\begin{aligned} Z[J] \; &= \; \langle 0|0\rangle = \int_C \mathcal{D}\phi \exp\left\{ -\int d^D s \right. \\ &\left. \times \left[\tfrac{1}{2}(\nabla\phi)^2 + \frac{g}{4n+2}\phi^{4n+2} - J\phi \right] \right\} \end{aligned}$$

$$G_n(x, y, z, \ldots) \equiv \frac{\delta^n}{\delta J(x)\delta J(y)\delta J(z)\cdots} \log Z[J]$$


φ^2
Im φ
90°
Re φ


φ^6
Im φ
30°
Re φ


φ^{10}
Im φ
18°
Re φ


φ^{14}
Im φ
12.8°
Re φ

Hermitian Hamiltonians: BORING!

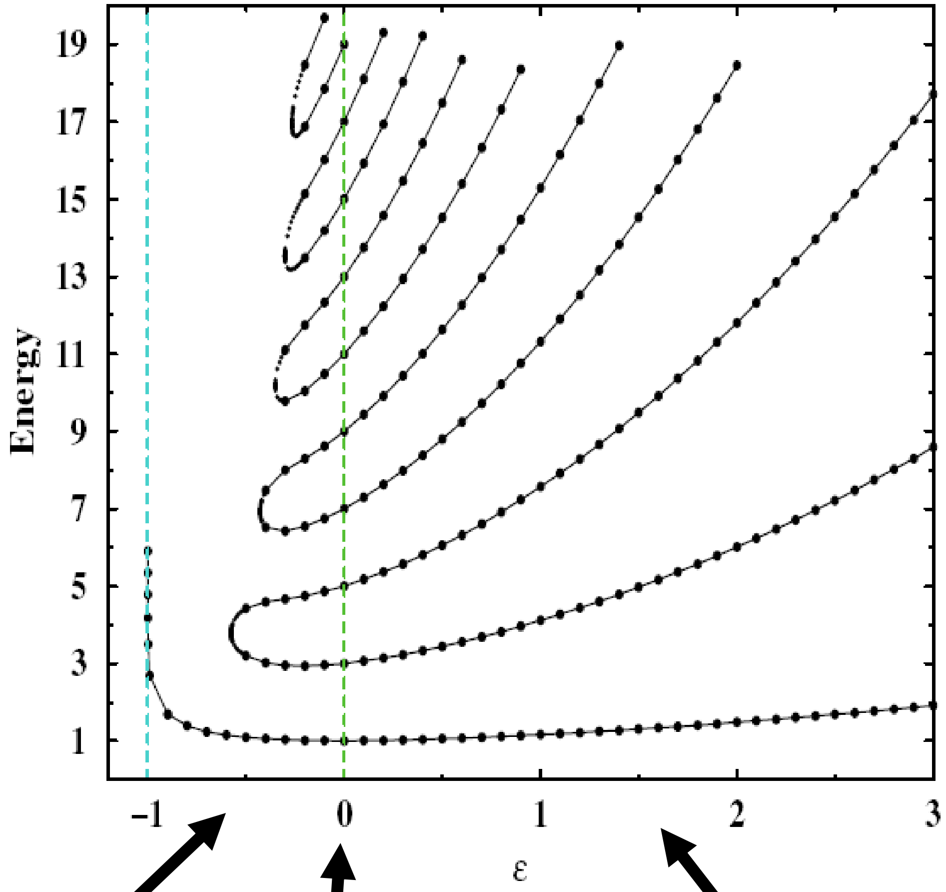
The eigenvalues are always real – nothing interesting happens



PT-symmetric Hamiltonians: ASTONISHING!

Phase transition between parametric regions of broken and unbroken *PT* symmetry...

$$H = p^2 + x^2(ix)^\epsilon \quad (\epsilon \text{ real})$$



Region of ***broken***
PT symmetry

***PT* Boundary**

Region of ***unbroken***
PT symmetry

Broken ***P***arro***T*** Unbroken ***P***arro***T***

Another example of a phase transition:

CMB and R. J. Kalveks, Int. J. Theor. Phys. **50**, 955 (2011)

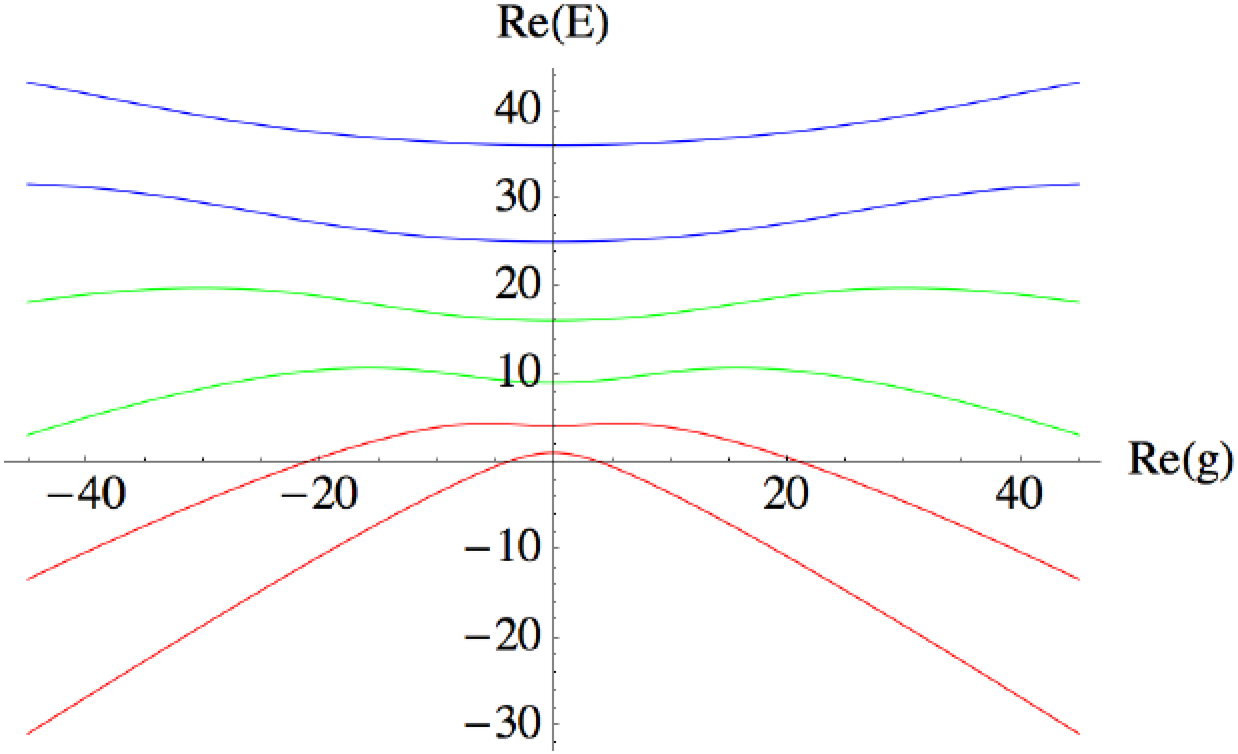
Replace the Heisenberg Algebra $[x, p] = 1i$ with the

E2 Algebra:

$$[u, J] = iv, \qquad [v, J] = -iu, \qquad [u, v] = 0$$

Hermitian Hamiltonian:

$$H = J^2 + gv$$



PT-symmetric Hamiltonian

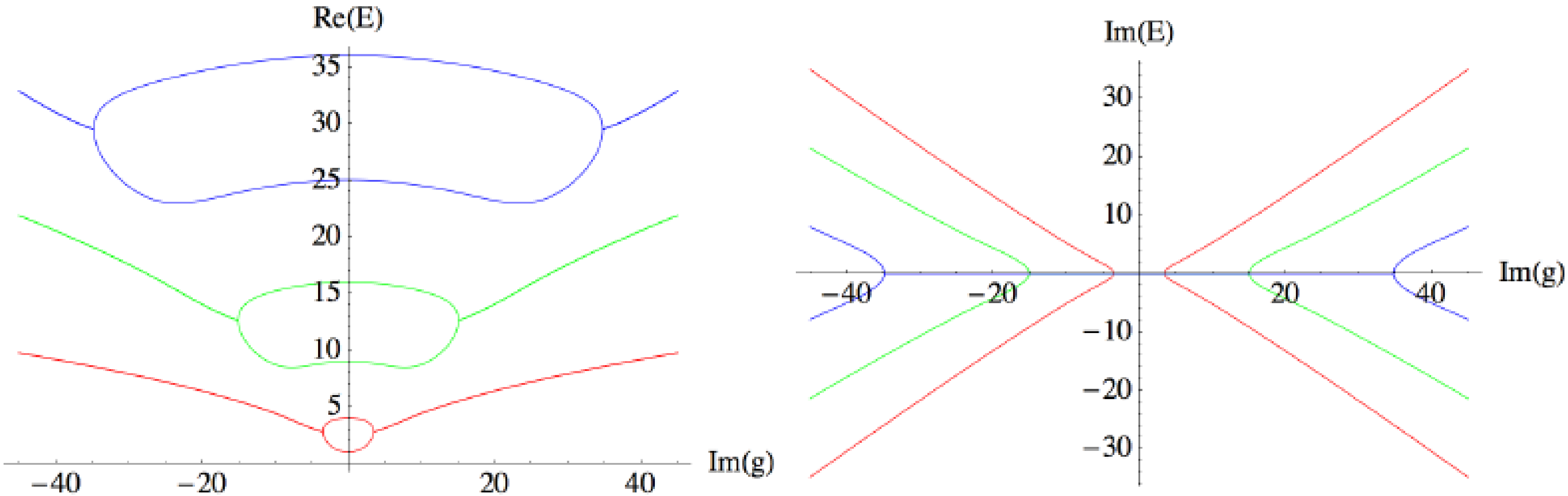
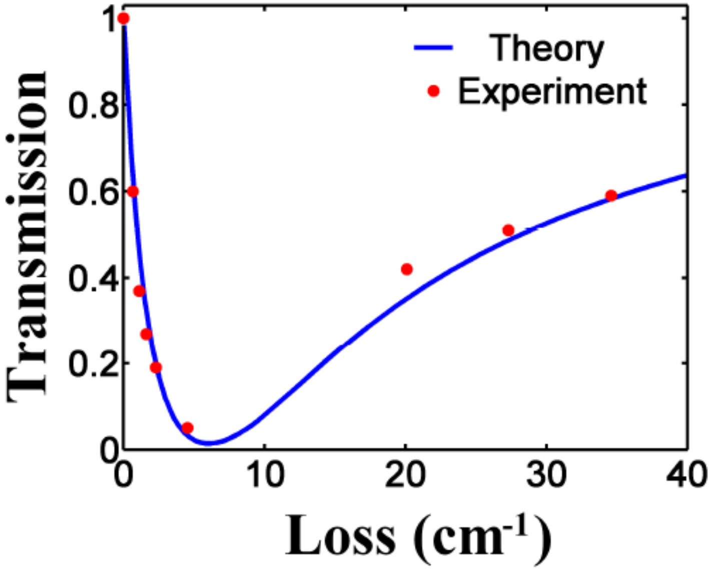


FIG. 3: Odd bosonic eigenvalues for the $\mathcal{PT}$-symmetric Hamiltonian (9) in which the parameter g is pure imaginary. The eigenvalues are plotted as functions of $\mathrm{Im}\, g$. The real (imaginary) parts of the eigenvalues are shown in the left (right) panel. Observe that the eigenvalues are all real when $-3.4645 < \mathrm{Im}\, g < 3.4645$; this is the region of unbroken $\mathcal{PT}$ symmetry. There is an infinite sequence of critical points; the next critical points are at $\mathrm{Im}\, g = \pm 15.0485$ and at ± 34.7994.

First observation of *PT* phase transition

Figure 4: Experimental observation of spontaneous passive $\mathcal{PT}$-symmetry breaking. Output transmission of a passive $\mathcal{PT}$ complex system as the loss in the lossy waveguide arm is increased. The transmission attains a minimum at 6 cm^{-1}.



A. Guo, G. J. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G. A. Siviloglou, and D. N. Christodoulides, *Physical Review Letters* **103**, 093902 (2009)

What exactly is this *PT* phase transition?

To understand the *PT* phase transition – introduce
PT *classical mechanics*

…a spin-off from ***PT***QM

Motion on the real axis

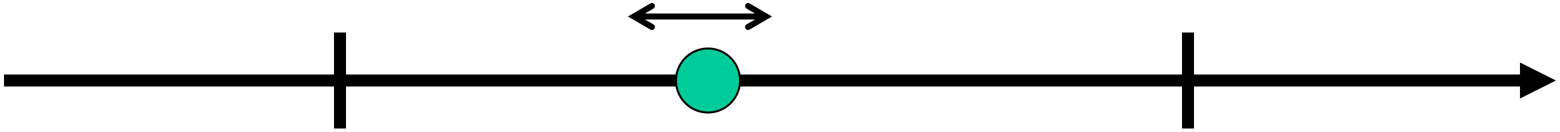
Motion of particles is governed by Newton's Law:

$$F=ma$$

In freshman physics this motion is restricted to the REAL AXIS.

Harmonic oscillator: Particle on a spring

Back and forth motion on the real axis:

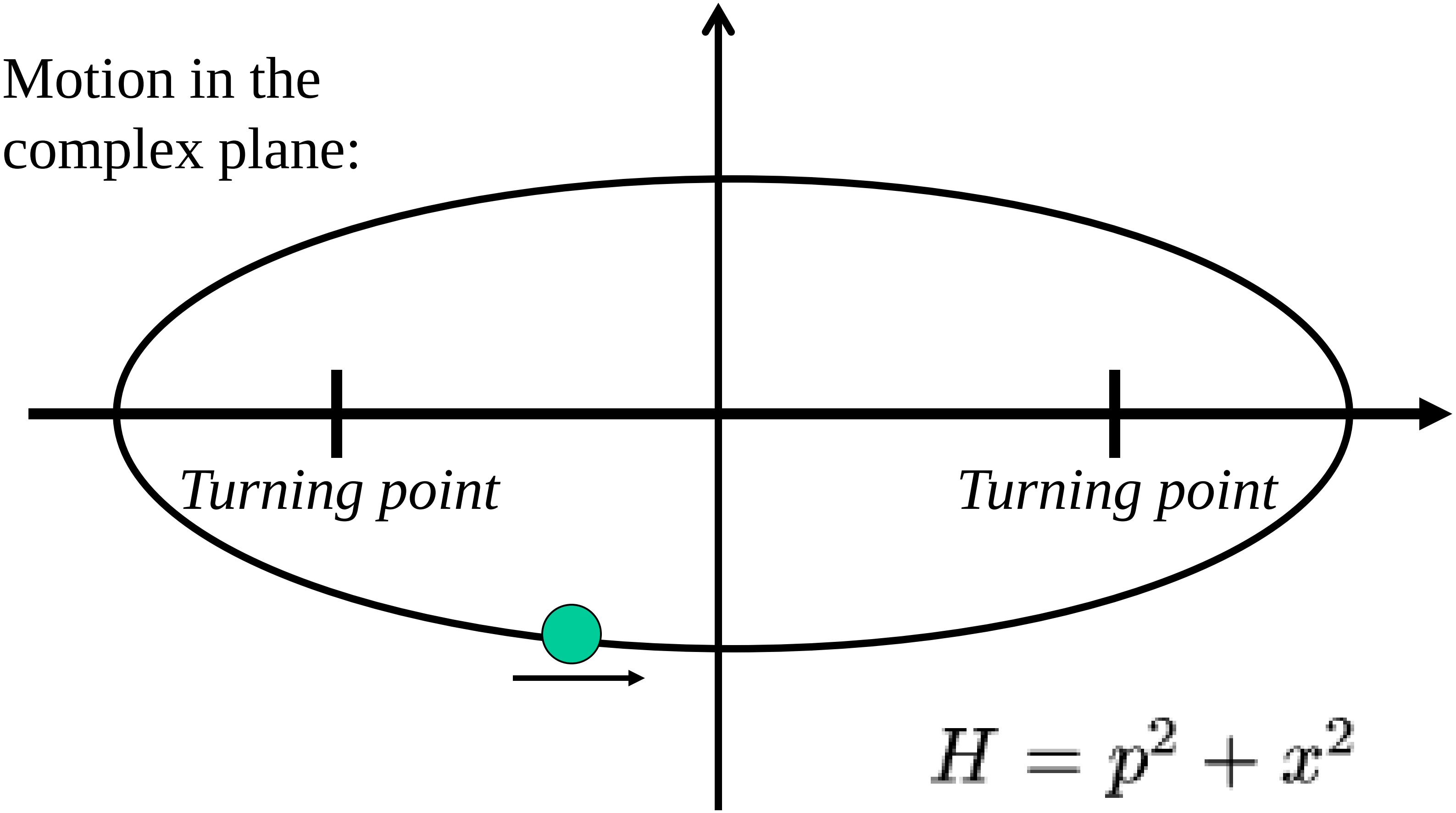


Turning point *Turning point*

$$H = p^2 + x^2 \qquad (\epsilon = 0)$$

Harmonic oscillator:

Motion in the complex plane:



Turning point

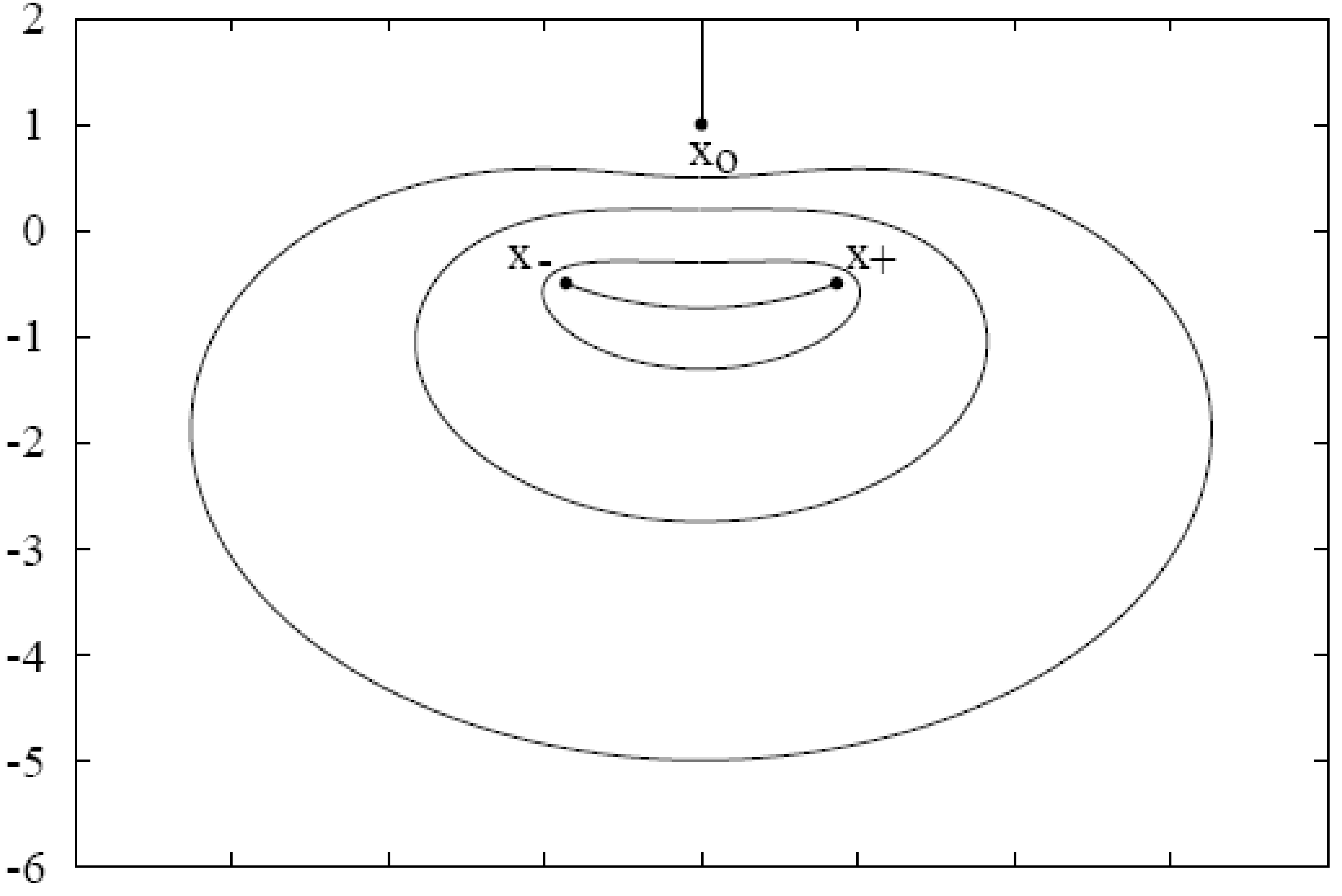
Turning point

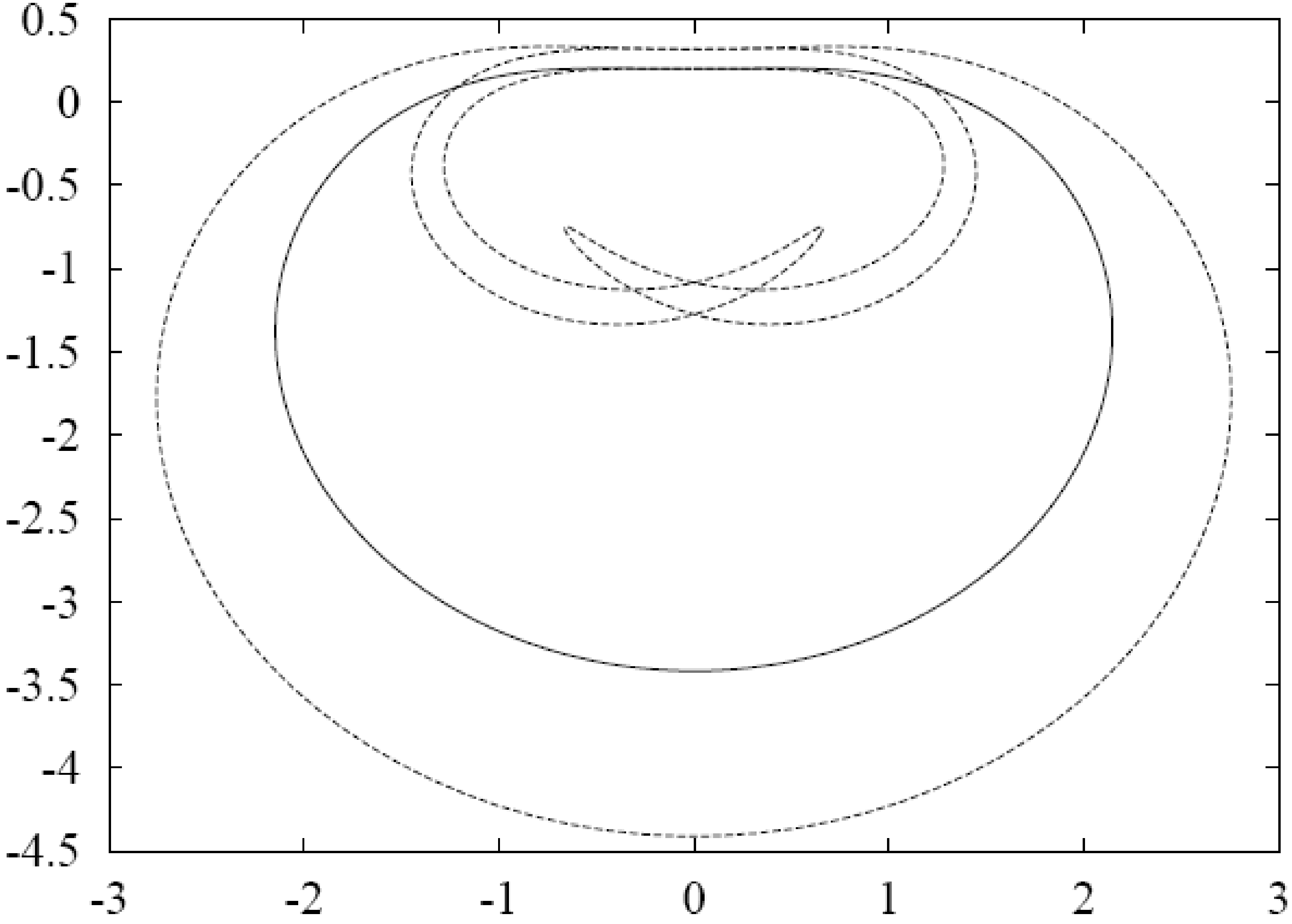
$$H = p^2 + x^2$$

(ϵ = 0)

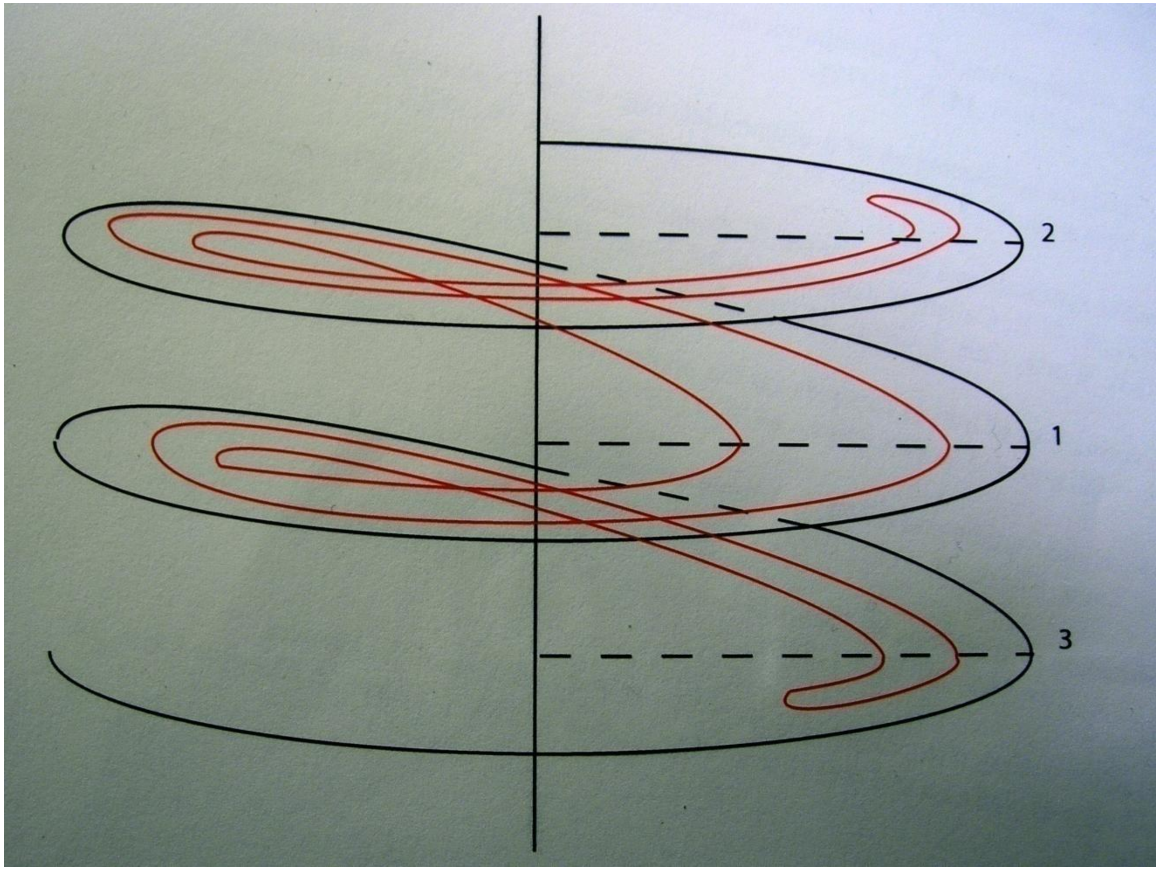
$$H = p^2 + ix^3$$

$(\epsilon = 1)$

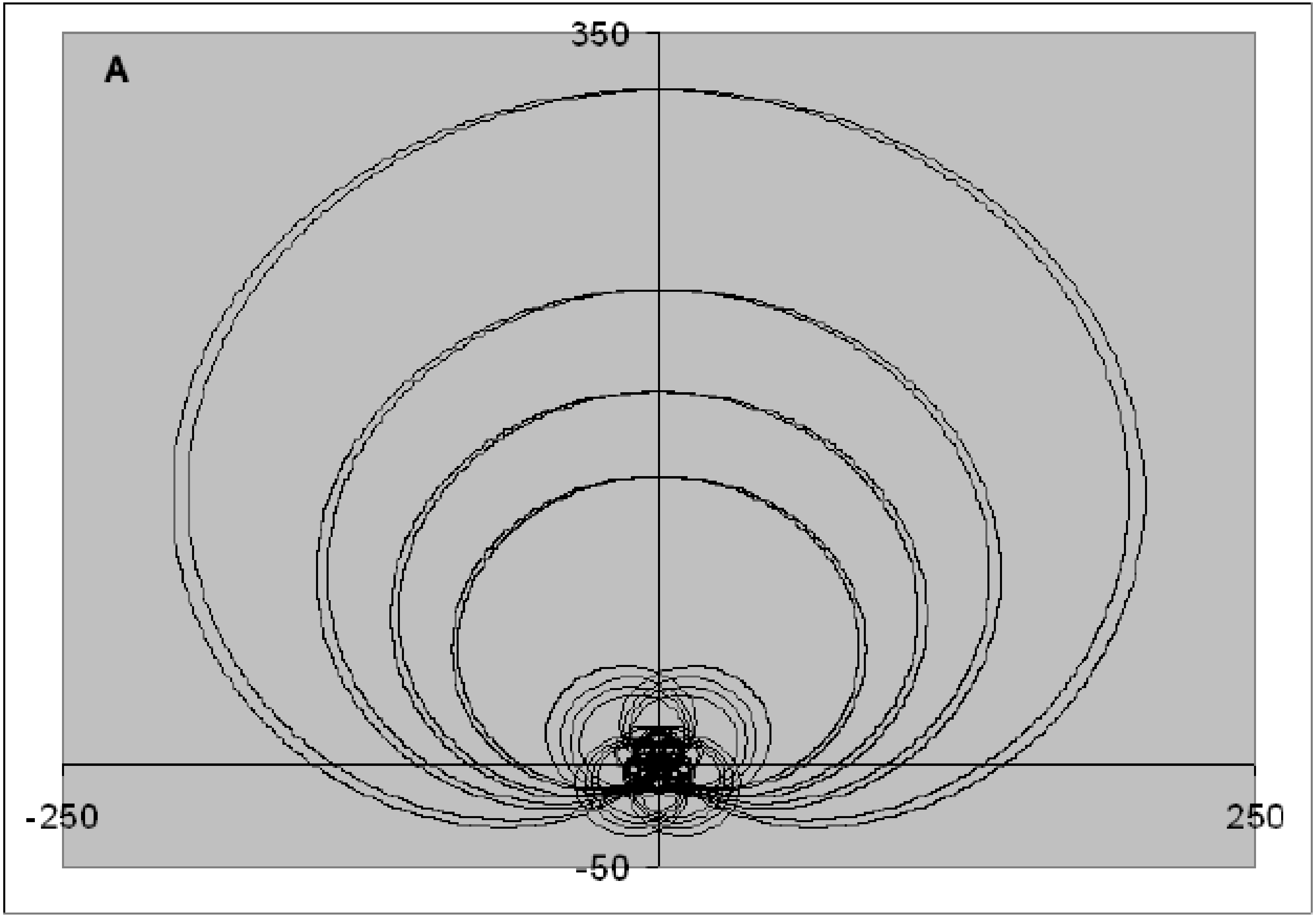




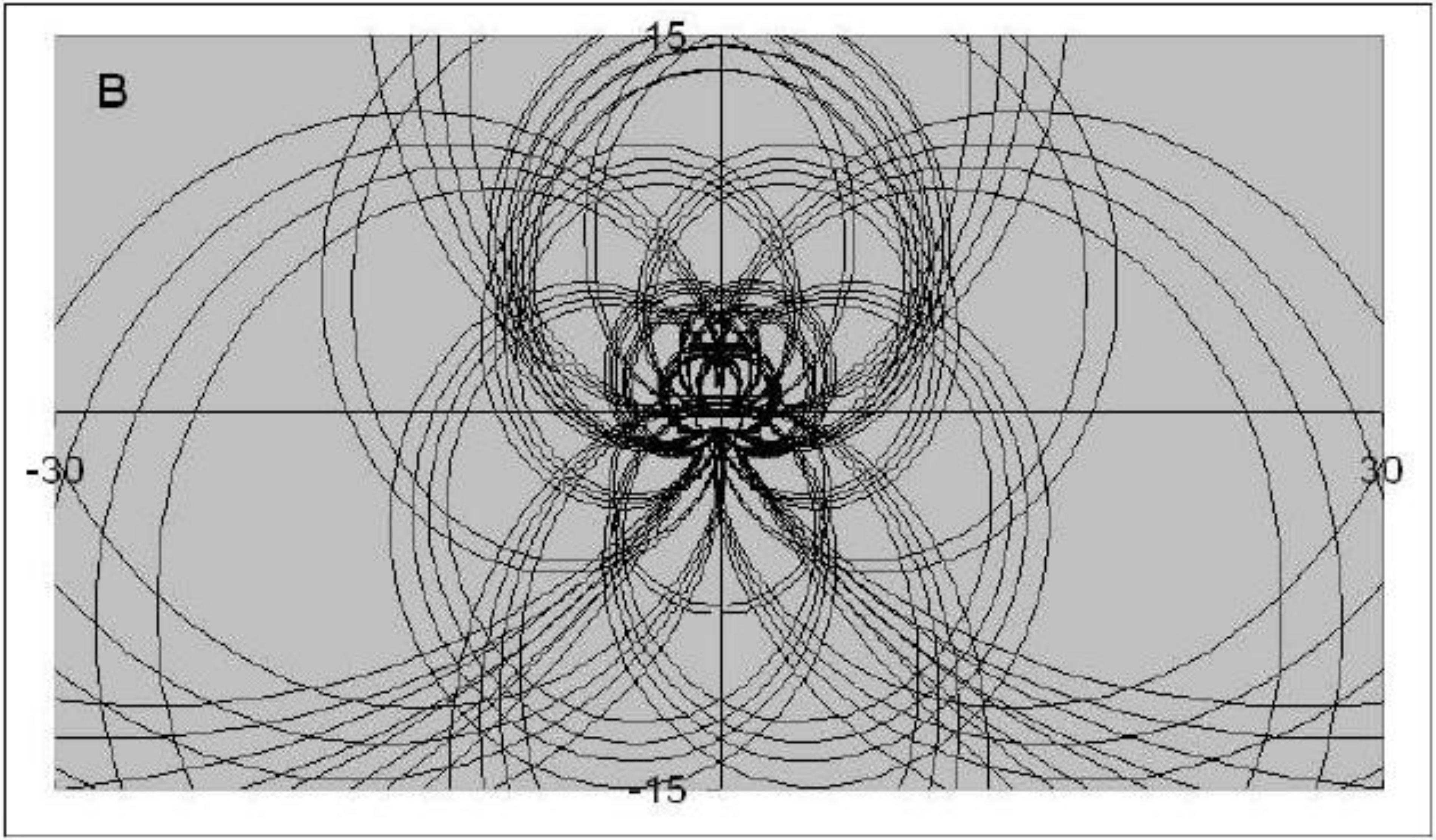
$\varepsilon = \pi - 2$



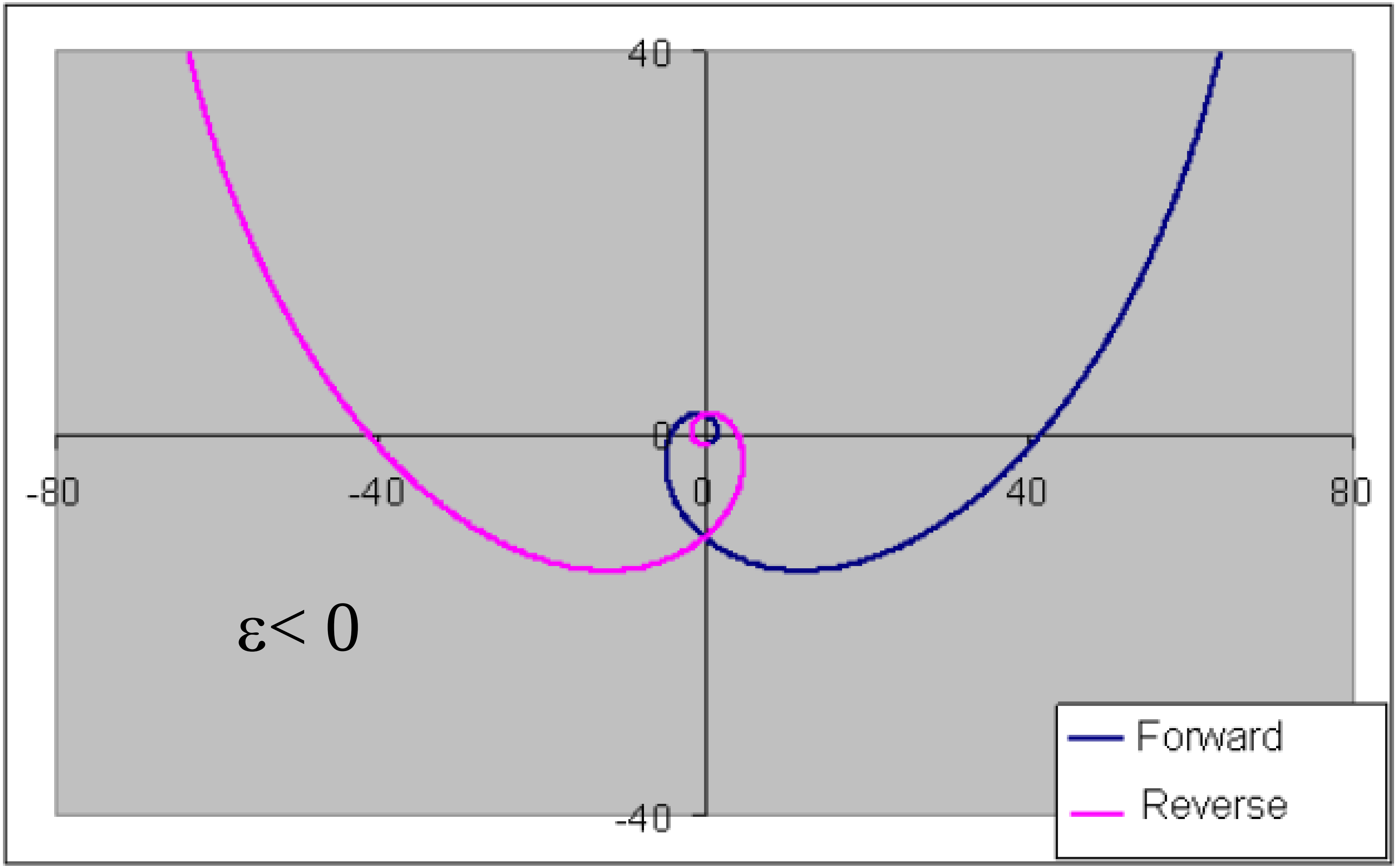
Classical orbit that visits three sheets of the Riemann surface



$\epsilon = \pi - 2$ 11 sheets


B
15
-30
30
-15

Broken *PT* symmetry – orbit not closed



$\varepsilon < 0$

Bohr-Sommerfeld Quantization of a complex atom

$$\oint dx\, p = \left(n + \frac{1}{2}\right)\pi$$

The effect of closed orbits for real energy vs. open orbits for complex energy suggests a way to understand tunneling...

CMB, D. C. Brody, and D. W. Hook, J. Phys. A **41**, 352003 (2008)

CMB and D. W. Hook, arXiv: hep-th/1011.0121

Quartic potential: REAL ENERGY

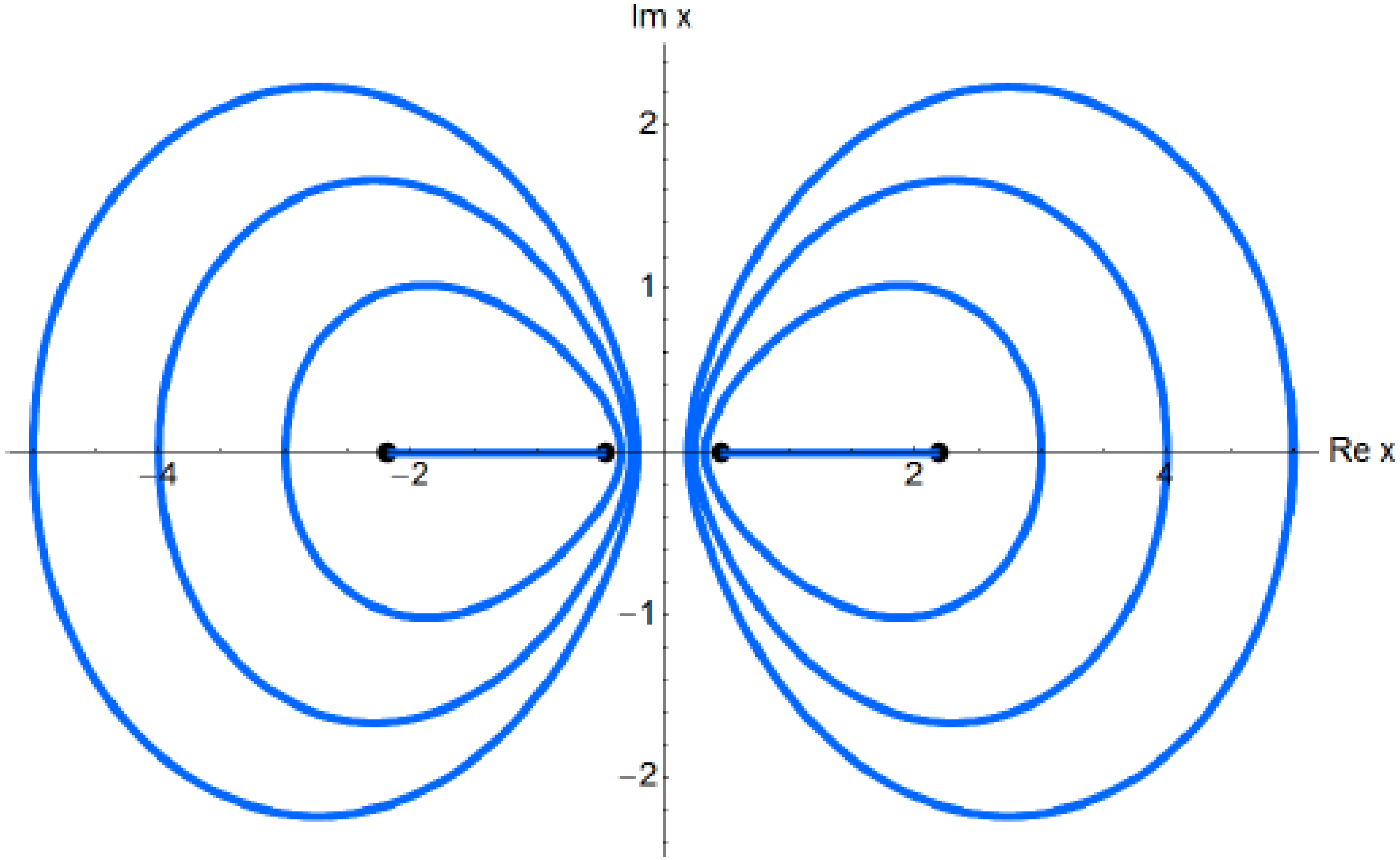


FIG. 1: Eight classical trajectories in the complex-x plane representing a particle of energy $E = -1$ in the potential $x^4 - 5x^2$. The turning points are located at $x = \pm 2.19$ and $x = \pm 0.46$ and are indicated by dots. Because the energy is real, the trajectories are all closed. The classical particle stays in either the right-half or the left-half plane and cannot cross the imaginary axis. Thus, when the energy is real, there is no effect analogous to tunneling.

COMPLEX ENERGY:

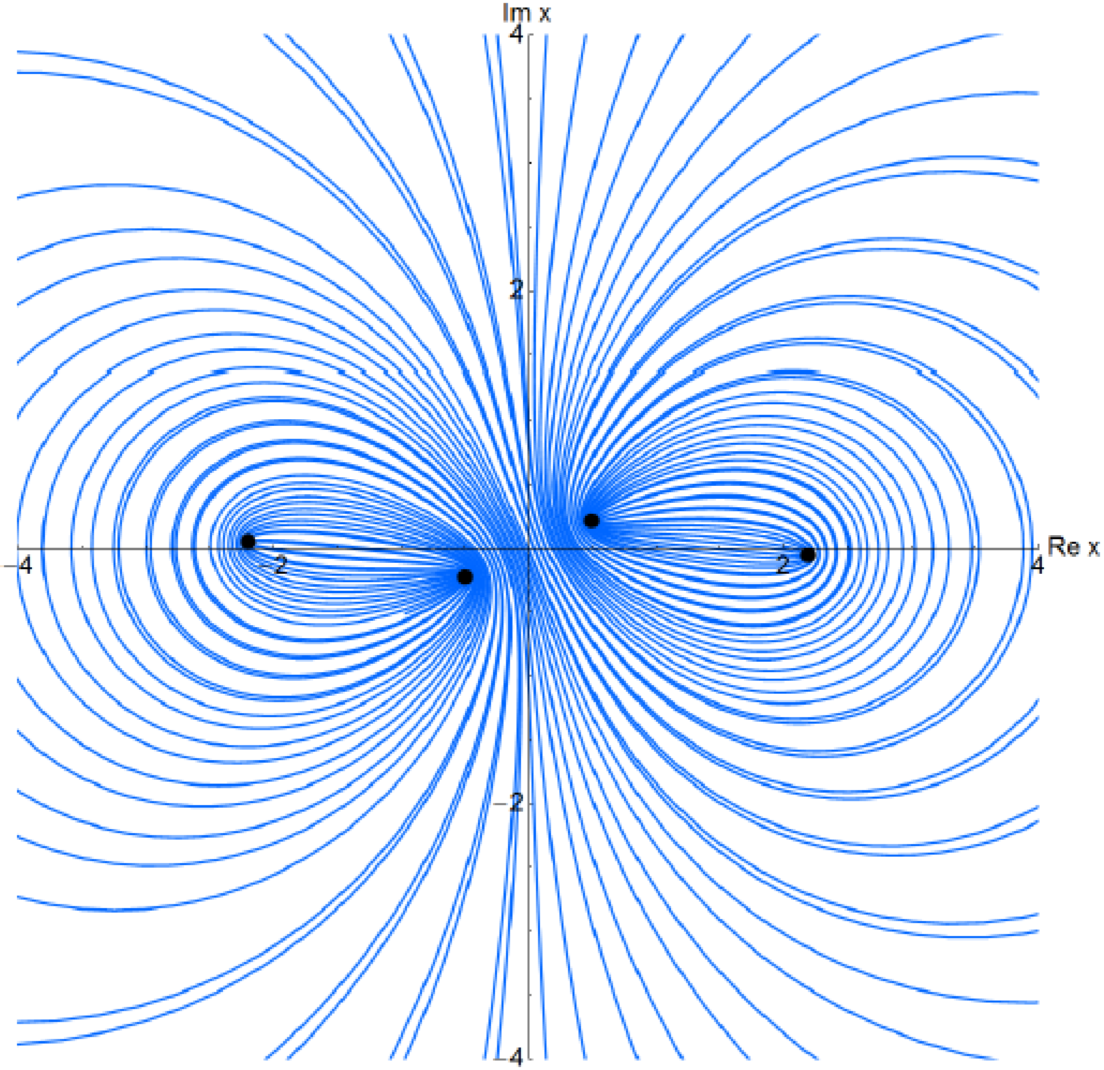
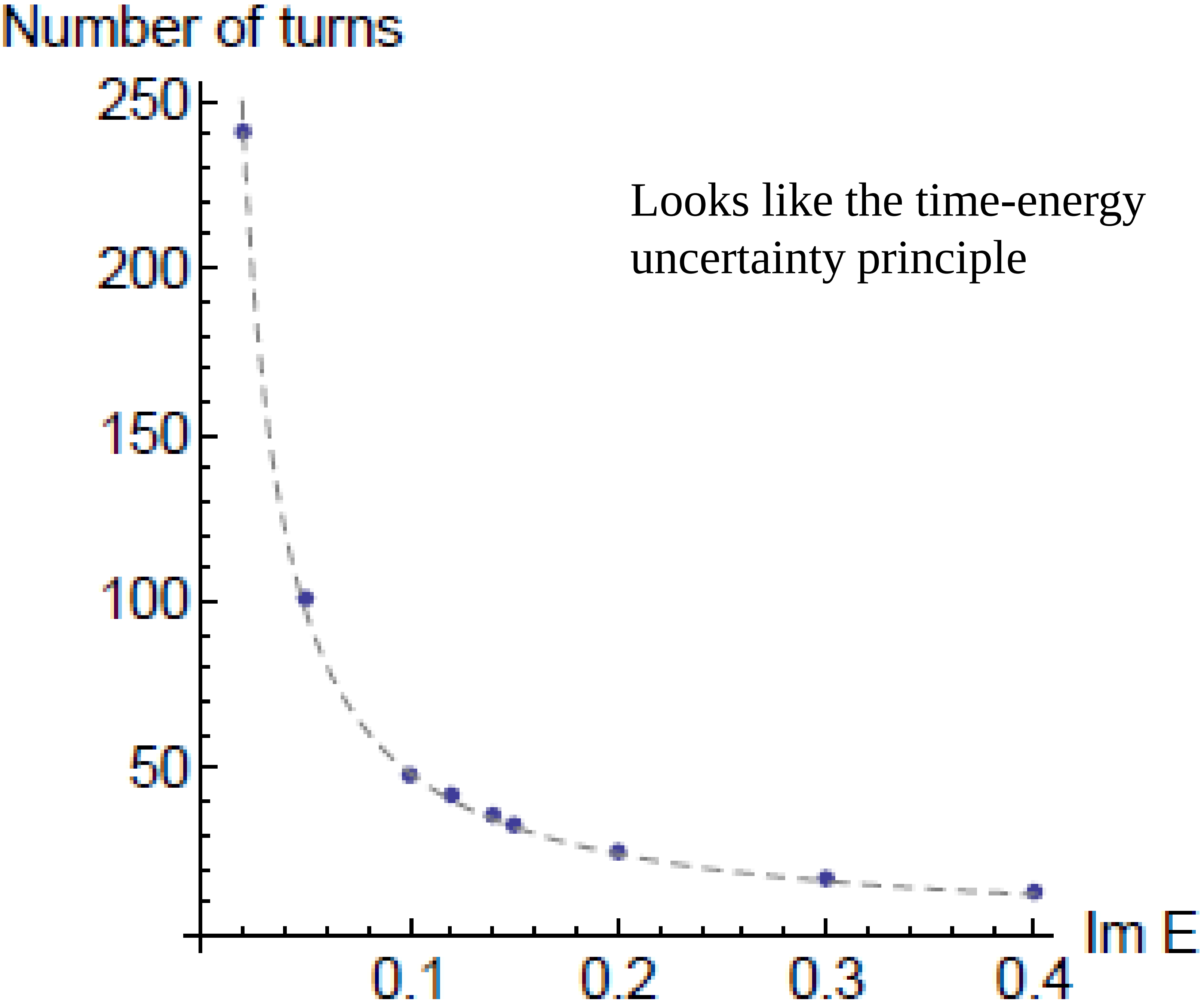


FIG. 2: Classical trajectory of a particle moving in the complex-x plane under the influence of a double-well $x^4 - 5x^2$ potential. The particle has complex energy $E = -1 - i$ and its trajectory does not close. The trajectory spirals outward around one pair of turning points, crosses the imaginary axis, and then spirals inward around the other pair of turning points. It then spirals outward again, crosses the imaginary axis, and goes back to the original pair of turning points.



Looks like the time-energy uncertainty principle

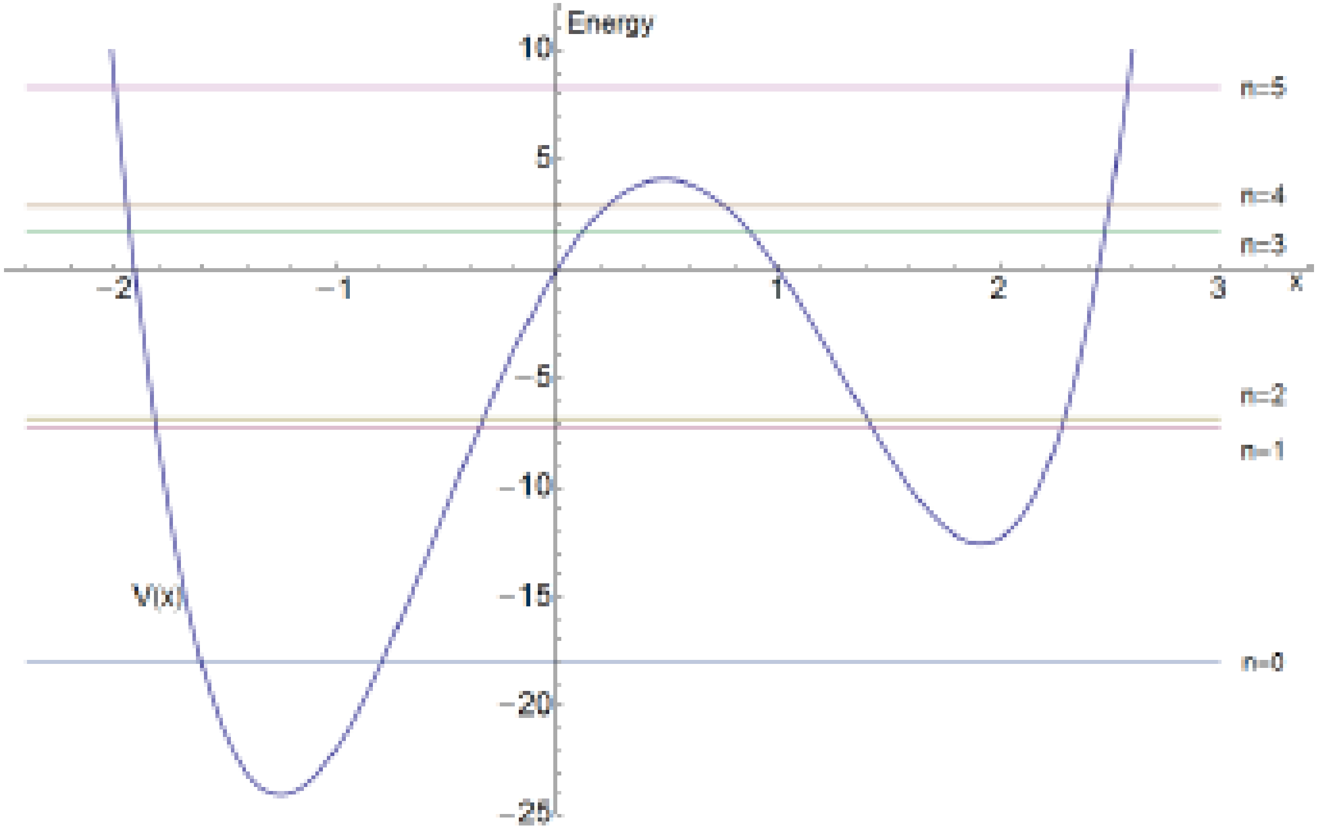


FIG. 1: Quartic asymmetric double-well potential $V(x)$ in (1) showing the first six quantum energy levels. The bottom of the left well is at $V = -24.0384$, and the bottom of the right well is at $V = -12.5501$. The ground-state energy E_0 lies below the bottom of the right potential well. The next four energy levels lie between the bottom of the right potential well and the top of the barrier, which is at $V = 4.1144$. The sixth energy level E_5 lies above the barrier.

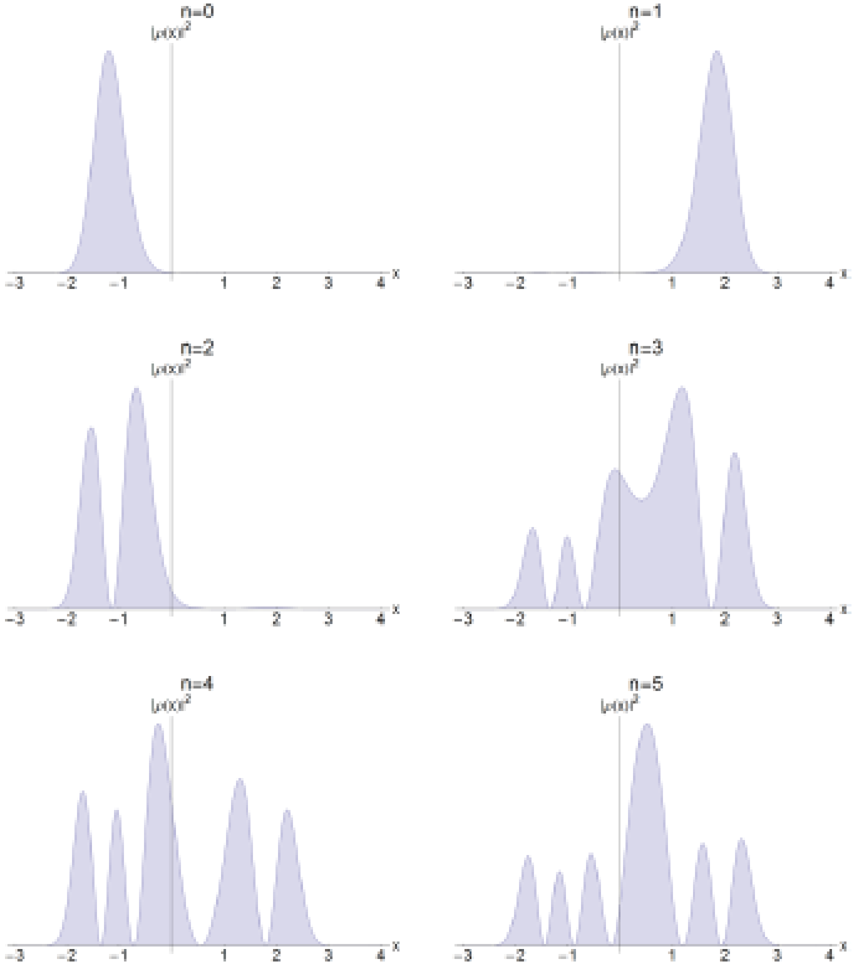


FIG. 2: Quartic asymmetric double-well potential $V(x)$ in (1) showing the first six quantum energy levels. The bottom of the left well is at $V = -24.0384$ and the bottom of the right well is at $V = -12.5501$. The ground-state energy E_0 lies below the bottom of the right potential well. The first four excited states lie between the bottom of the right potential well and the top of the barrier, which is at $V = 4.1144$. The fifth excited state E_5 lies above the barrier.

Periodic potential

CMB and T. Arpornthip,
Pramana **73**, 375 (2009)

$$H(x,p)=\frac{1}{2}p^2-\cos(x)$$

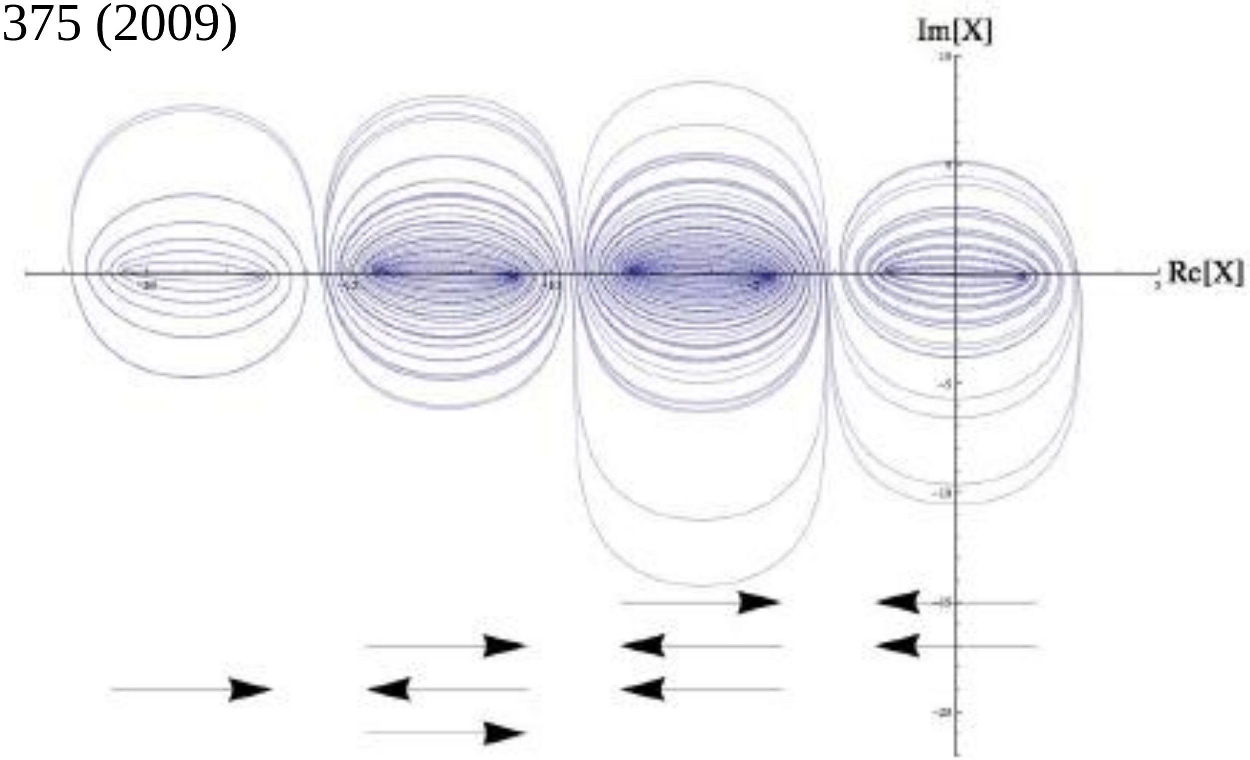


Figure 5. A tunneling trajectory for the Hamiltonian (2) with $E=0.1-0.15i$. The classical particle hops at random from well to well in a random-walk fashion. The particle starts at the origin and then hops left, right, left, left, right, left, left, right, right. This is the sort of behavior normally associated with a particle in a crystal at an energy that is not in a conduction band. At the end of this simulation the particle is situated to the left of its initial position. The trajectory never crosses itself.

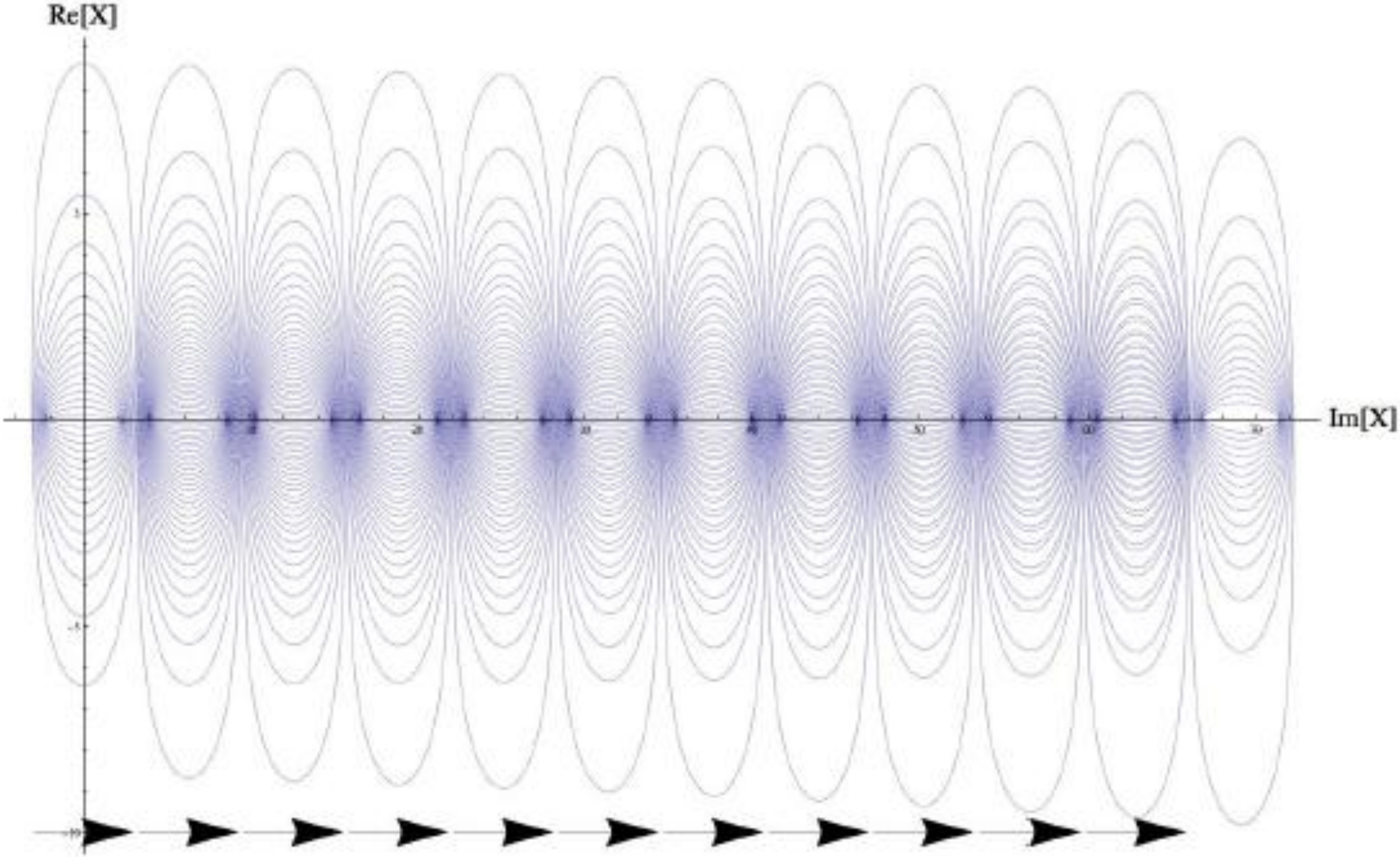


Figure 6. A classical particle exhibiting a behavior analogous to that of a quantum particle in a conduction band that is undergoing resonant tunneling. Unlike the particle in Fig. 5, this classical classical particle tunnels in one direction only and drifts at a constant average velocity through the potential.

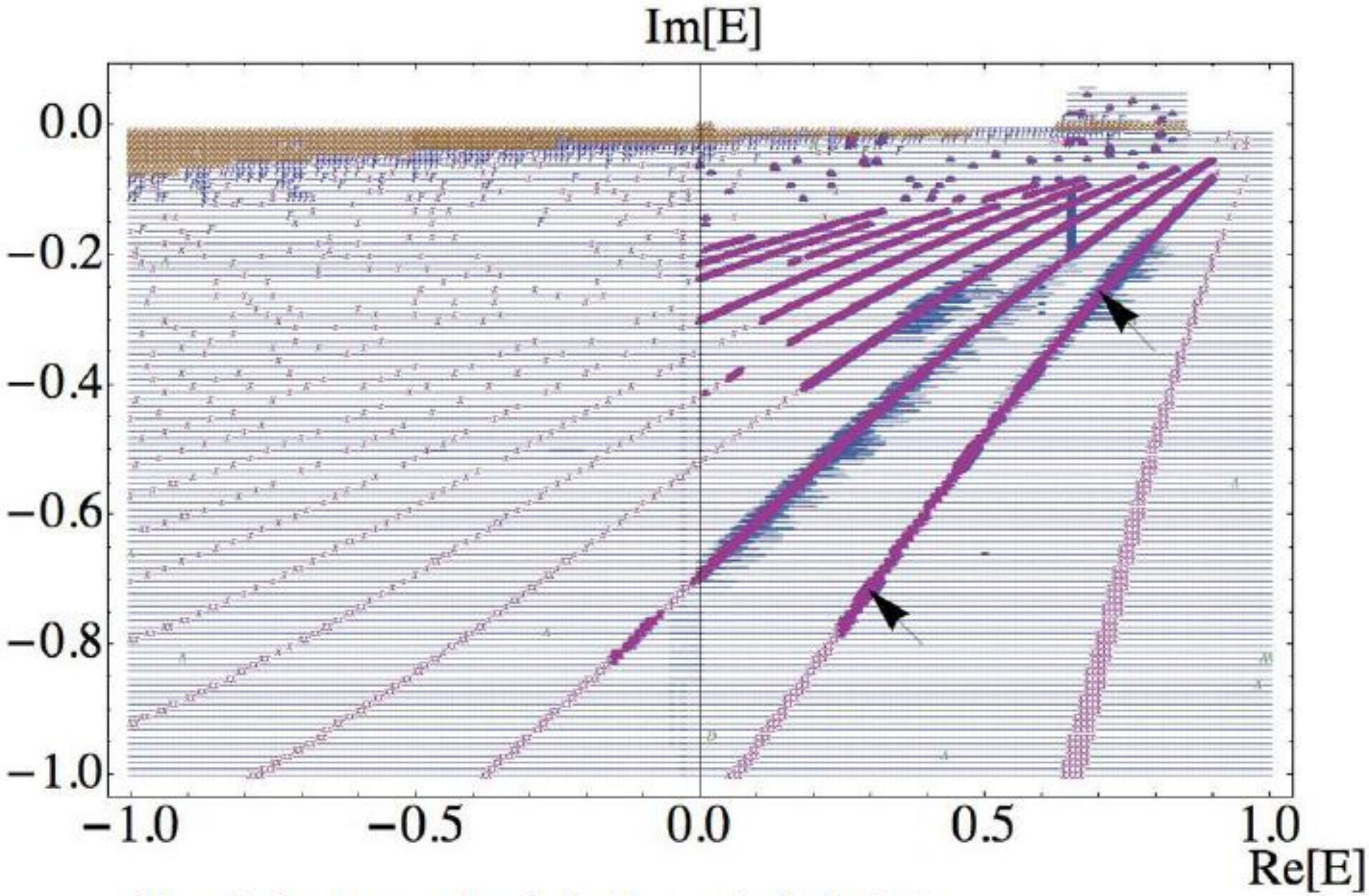


Figure 7. Complex-energy plane showing those energies that lead to tunneling (hopping) behavior and those energies that give rise to conduction. Hopping behavior is indicated by a hyphen - and conduction is indicated by an X. The symbol & indicates that no tunneling takes place; tunneling does not occur for energies whose imaginary part is close to 0. In some regions of the energy plane we have done very intensive studies and the X's and -'s are densely packed. This picture suggests the features of band theory: If the imaginary part of the energy is taken to be -0.9, then as the real part of the energy increases from -1 to $+1$, five narrow conduction bands are encountered. These bands are located near $\mathrm{Re}\,E = -0.95, -0.7, -0.25, 0.15, 0.7$. This picture is symmetric about $\mathrm{Im}\,E = 0$ and the bands get thicker as $|\mathrm{Im}\,E|$ increases. A total of 68689 points were classified to make this plot. In most places the resolution (distance between points) is 0.01, but in several regions the distance between points is shortened to 0.001. The regions indicated by arrows are blown up in Figs. 8 and 9.

But... It’s not so simple...
Complex energy does not always mean open orbits!

A. Anderson, CMB, U. Morone, arXiv: math-ph 1102.4822

Potential: $V(x) = x^4 - 5x^2$

Equation of motion: $[x'(t)]^2 + V(x) = E$

Solution: $x(t) = a\,\mathrm{sn}(ibt, k)$ (Jacobi elliptic function)

$$u = \int_0^{\mathrm{sn}(u,k)} \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}} \qquad k^2 = \frac{a^2}{b^2} = \frac{5-\sqrt{25+4E}}{5+\sqrt{25+4E}}$$

Trajectory closes under the replacement:

$$ibt \to ibt + 4mK(k) + 2niK(k')$$

Condition for having a periodic orbit: $$\frac{n}{m} = \frac{\mathrm{Im}[2iK(k)/b]}{\mathrm{Im}[K(k')/b]}$$

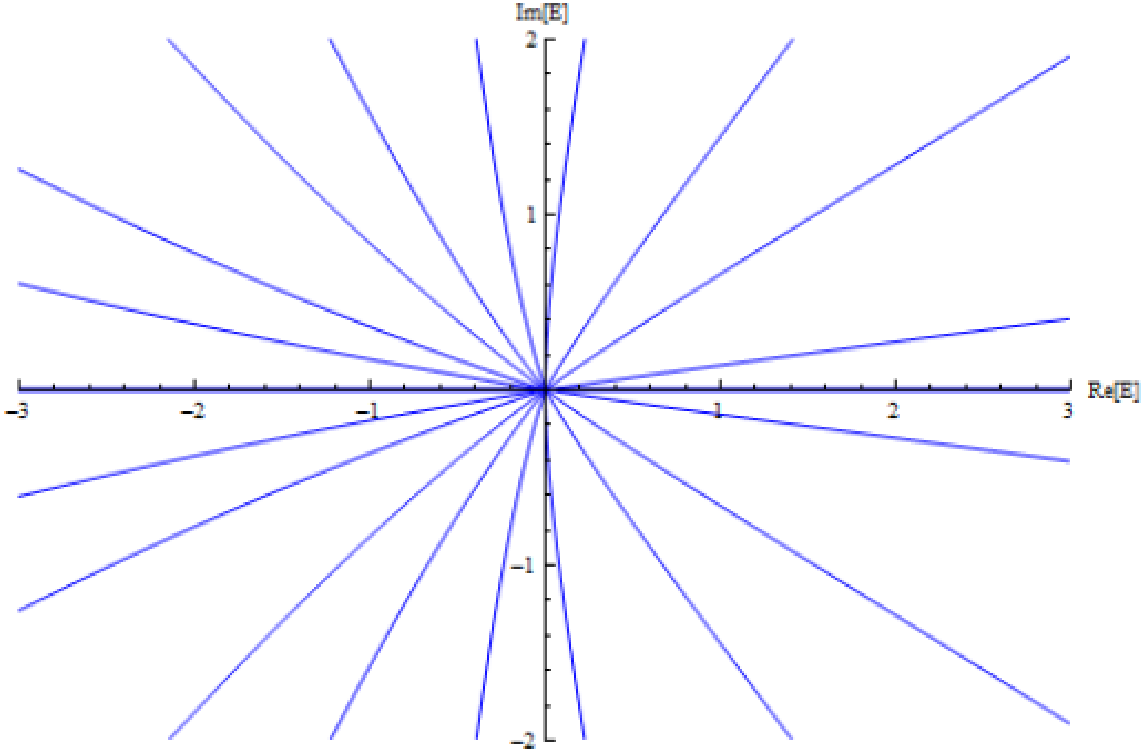
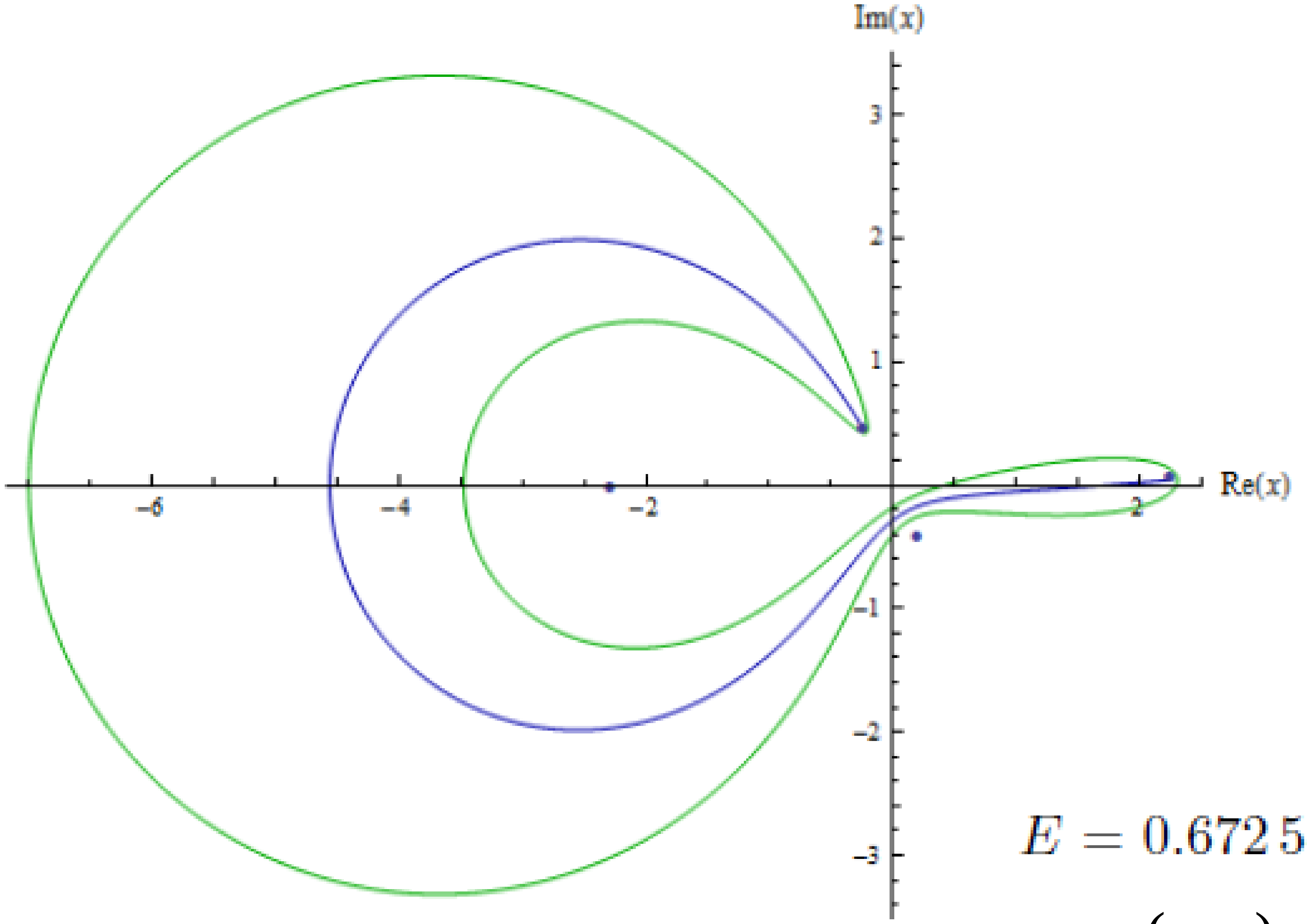
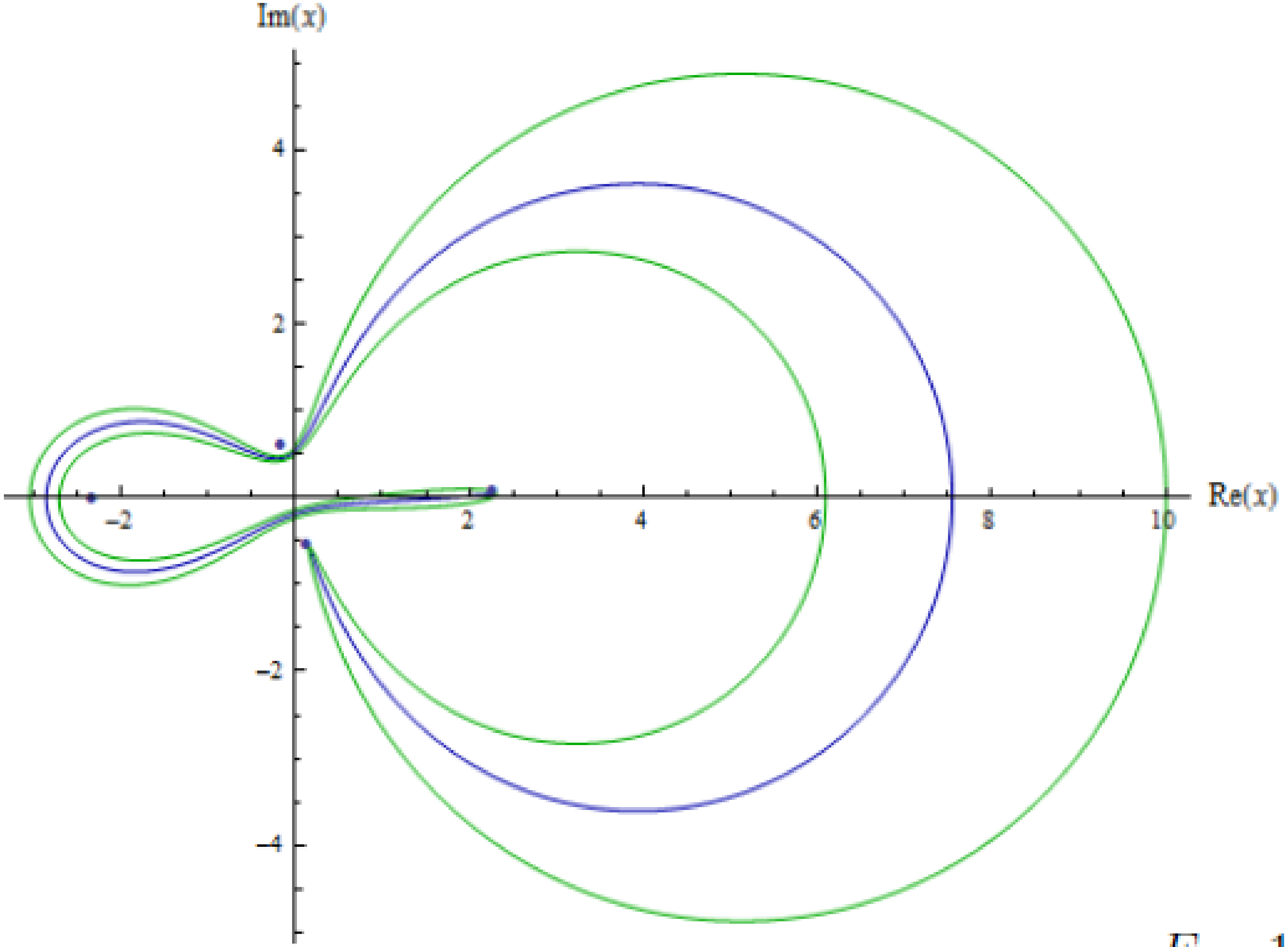


FIG. 3: Some quantized complex energies E for the potential $V(x) = x^4 - 5x^2$ for $-3 < \mathrm{Re}\, E < 3$ and $-2 < \mathrm{Im}\, E < 2$ [see (14)]. These curves represent some of the (infinite number of) special complex energies E for which the classical orbits are periodic. These energies occur for rational values of $n/m \geq 2$. When $n = 2$ and $m = 1$, E is real and positive. (This corresponds to oscillatory particle motion above the barrier in the potential.) The energy curve just above the positive-real axis in this figure corresponds to $(n, m) = (5, 2)$. Subsequent energy curves in anticlockwise order correspond to $(n, m) = (3, 1)$, $(n, m) = (4, 1)$, $(n, m) = (5, 1)$, $(n, m) = (7, 1)$, $(n, m) = (10, 1)$, $(n, m) = (20, 1)$, $(n, m) = (40, 1)$, and the negative real axis corresponds to $n/m = \infty$. (When $E < 0$, the particle motion is oscillatory and confined to either the left or the right well.) The energy curves in the lower-half E plane are complex conjugates of the energy curves in the upper-half E plane. Near the origin these curves are asymptotically straight lines [see (15 and (16)].



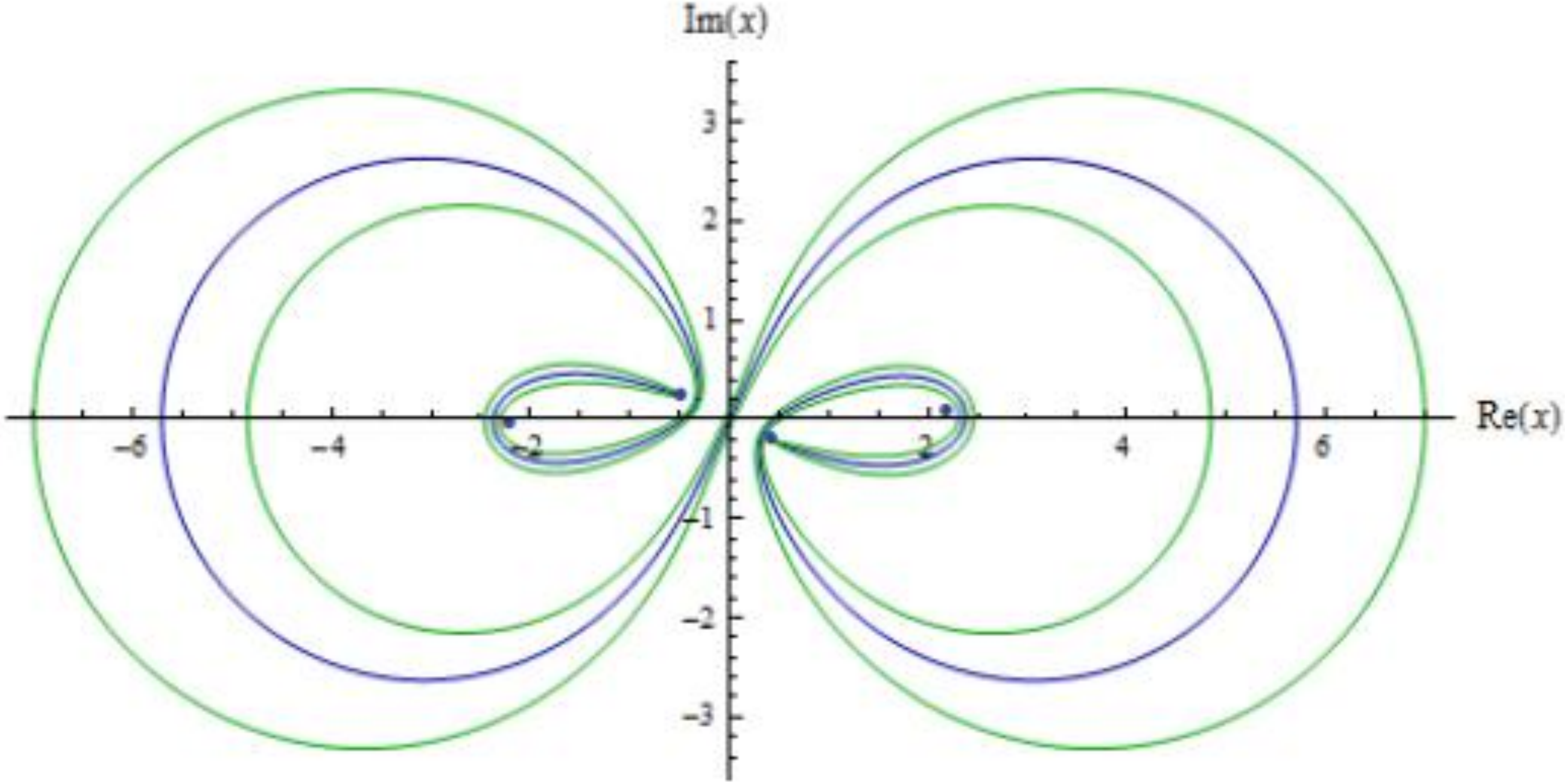
$$E = 0.672\,543\,108\,9 + i$$

$(n,m) = (3,1)$



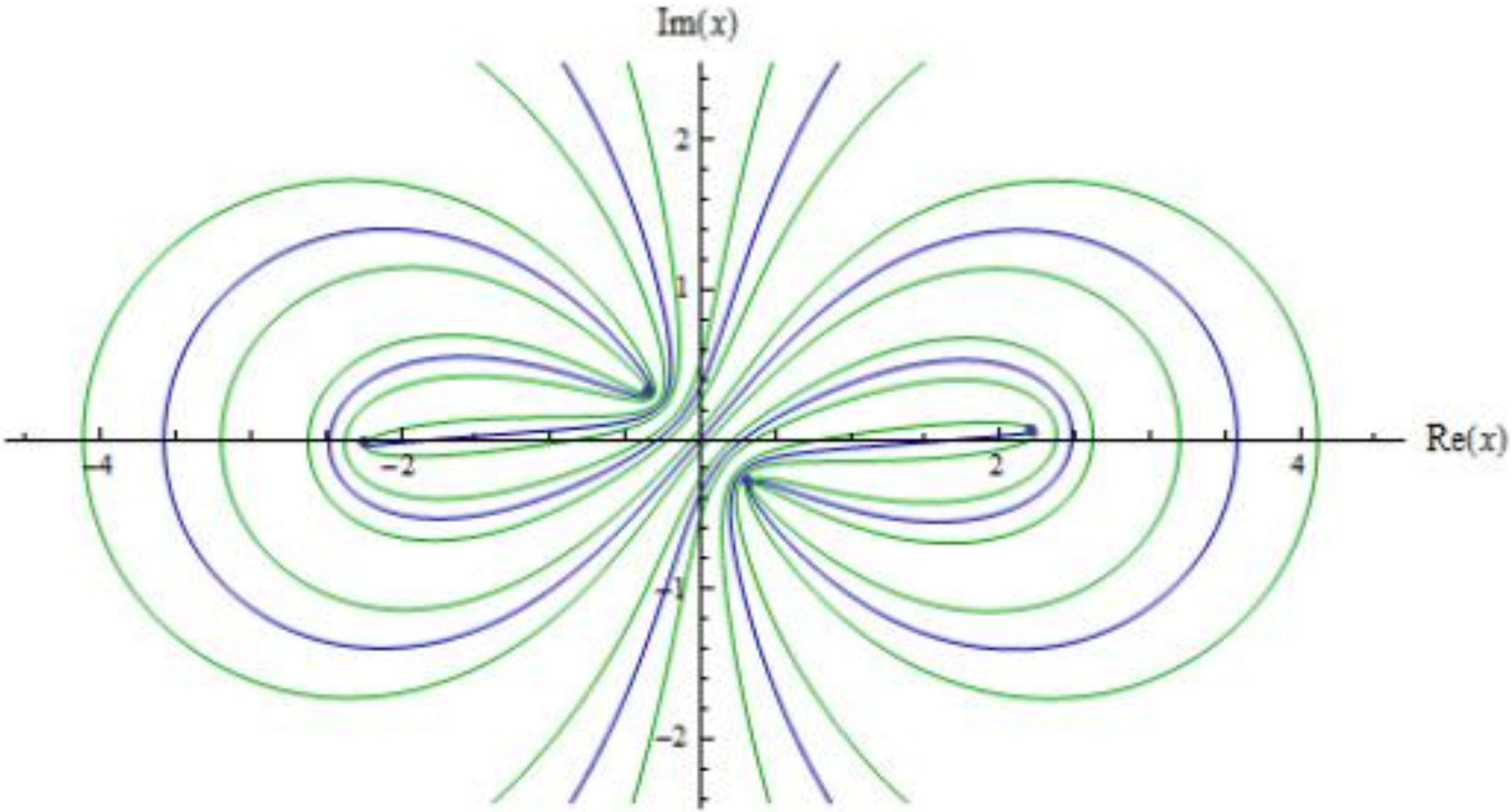
$E = 1.540\,288\,094\,6 + i$

$(n,m) = (5,2)$



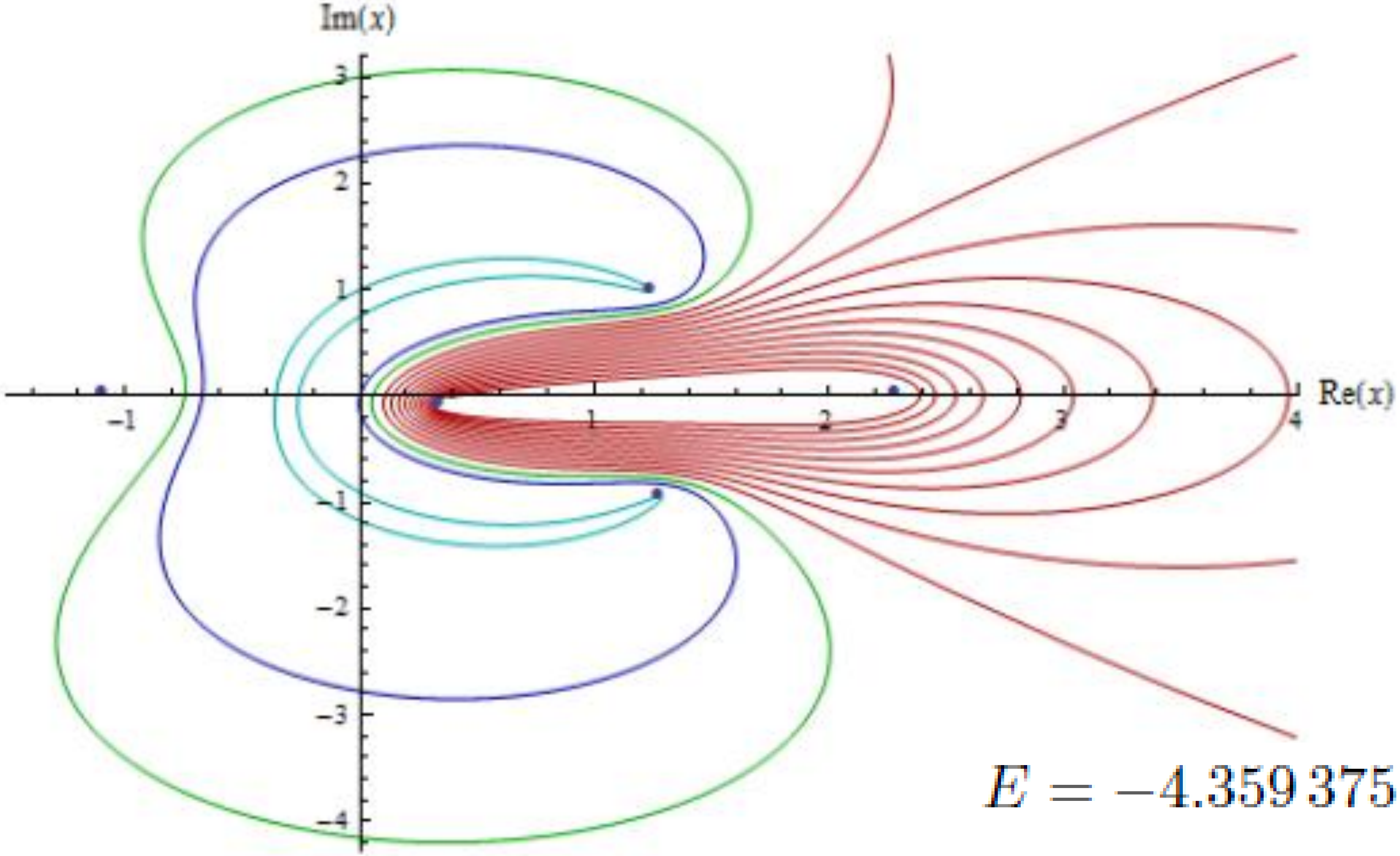
$$E = -0.852\,958\,824\,6 + i$$

$(n,m) = (8,1)$



$E = -0.144\,984\,595\,5 + i$

$(n,m) = (14,3)$

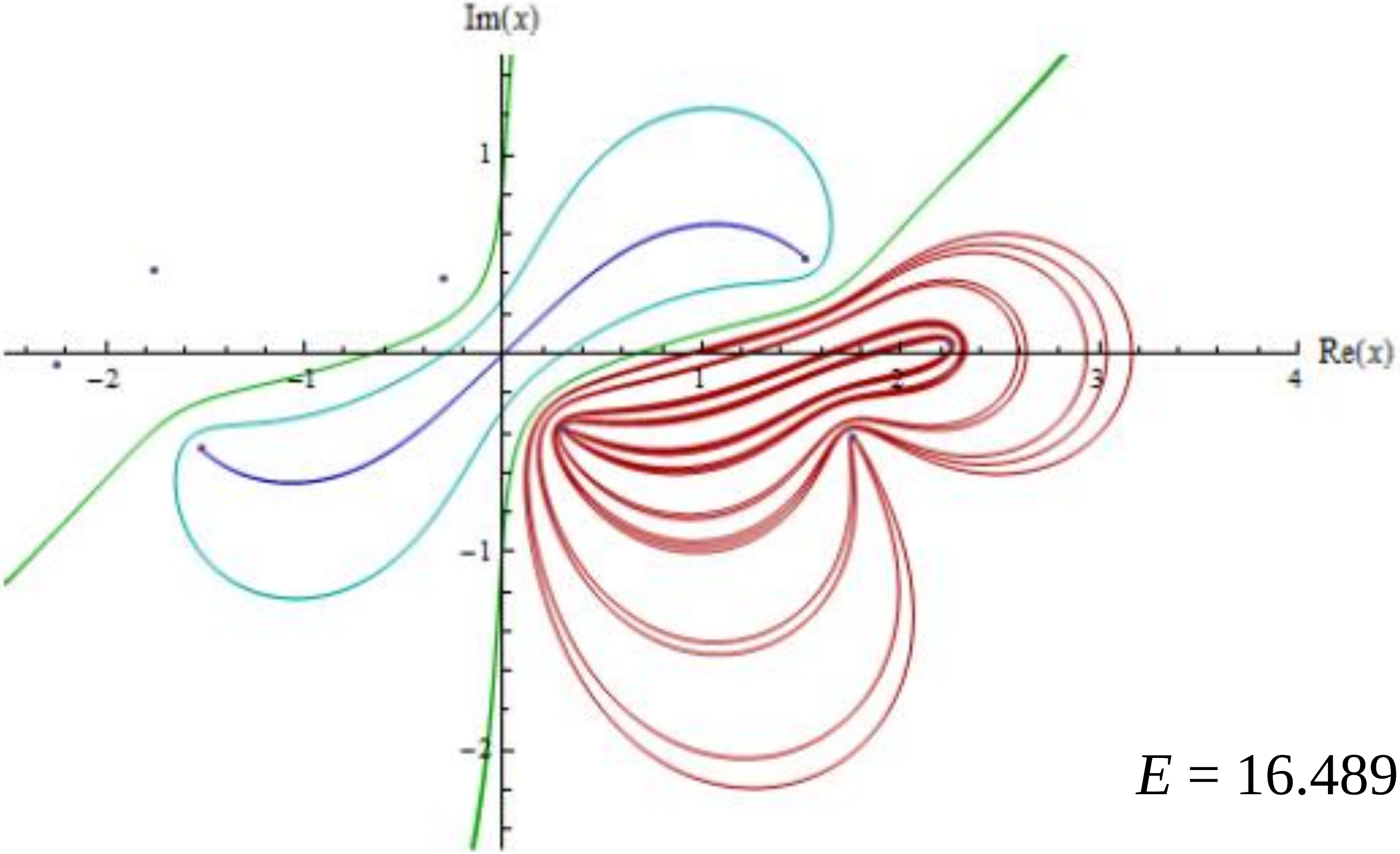


$$E = -4.359\,375 + i$$

Periodic (blue, cyan, green) and nonperiodic (red) trajectories for the sextic potential

$$V(x) = x^6 - 5x^5 - 4x^4 + 11x^3 - \tfrac{11}{4}x^2 - 13x$$

(Separatrix not shown)



$E = 16.489 + 10i$

$$V(x) = (x-1)^2(x+1)^2(x-2)^2(x+2)^2$$

Periodic trajectories (blue, cyan) and nonperiodic trajectory (red)
Separatrix curve (green)

Thanks for listening!


Pt
Lake
TRAILHEAD
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