Non-hermitian scattering and non-linear bifurcations with Bose-Einstein condensates

Joachim Brand
Two Professor positions open in the areas of biological, physical, mathematical sciences. Looking for *distinguished performance in research*.

PhD scholarships and postdoctoral scholarships available. Alexander von Humboldt (Feodor Lynen) scholars welcome.
Quantum gases at Massey

- Few-particles, full quantum dynamics
- Soliton dynamics
- Superfluid Fermi gases: BEC-BCS crossover
- Low dimensional BEC Quantum fluctuations and strong correlations
- Macroscopic quantum superpositions in strongly-interacting systems
- Josephson Junction analogs, Vortex tunneling

Jake, David, Joachim, Thomas, Renyuan, Gabriele, Oleksandr
Outline

• Introduction

• Non-Hermitian physics in quasiparticle scattering
  – Levinson’s theorem
  – Fano Resonance

• Bifurcations and instabilities
  – fluxon-dark soliton bifurcation as exceptional point
Ultra-cold atom experiments

Lene Hau
Harvard
Na BEC

University of Otago (New Zealand)
Andrew Wilson - Nils Kaergard
Rb - K experiment
The many-body physics

**Topics:**
- Strongly correlated (quantum) phases in optical lattice potentials
- Artificial gauge potentials
- Few-particle (Efimov) physics
- Dipolar (long-range, anistropic) interactions
- Macroscopic quantum phenomena
- Nonlinear waves

*Interactions* are typically short range (van-der Waals) and *can be tuned* by exploiting magnetic-field dependent Fano-Feshbach resonances.

**Superfluid Fermi gas**

Picture credits: Jin group

Picture credits: Ketterle group

Thursday, 30 June 2011
Scattering to probe BECs?

What can we learn from scattering atoms identical to the BEC atoms? Are there interesting coherent effects?

BECs are not created at zero temperature. Thermal atoms (at low density) would scatter from BEC.

Is quasiparticle scattering different from single-particle QM scattering theory?
Theory: Bose-Einstein Condensate (BEC)

Bose gas in an external potential

\[ i\hbar \frac{\partial}{\partial t} \tilde{\Psi}(\mathbf{r}, t) = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}}(\mathbf{r}, t) + \int \tilde{\Psi}^\dagger(\mathbf{r'}, t)V(\mathbf{r'} - \mathbf{r})\tilde{\Psi}(\mathbf{r'}, t)d\mathbf{r'} \right] \tilde{\Psi}(\mathbf{r}, t) \]

For BECs we may use the classical or mean field (Hartree) approximation:

**Gross-Pitaevskii (GP) equation**

\[ i\hbar \frac{\partial}{\partial t} \psi(\mathbf{r}, t) = \left[ -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}}(\mathbf{r}, t) + \frac{4\pi a_s}{m} |\psi(\mathbf{r}, t)|^2 \right] \psi(\mathbf{r}, t) \]

\[ a_s \quad \text{s-wave scattering length} \]

The GP equation is a **nonlinear Schrödinger equation** it supports dynamically stable and unstable solutions, soliton solutions, bifurcations, etc.
Elementary Excitations of the BEC

Time-dependent **Gross-Pitaevskii** equation

\[ i\hbar \frac{\partial}{\partial t} \psi(\vec{r}, t) = \left\{ -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{trap}} + \frac{4\pi a_s N}{m} |\psi|^2 \right\} \psi \]

Linearize the time-dependent equation around the stationary mean field:

\[ \psi(\vec{r}, t) = \phi(\vec{r}) e^{i\mu t/\hbar} + \delta\psi(\vec{r}, t) \]

\[ \delta\psi(\vec{r}, t) = u(\vec{r}) e^{i\varepsilon t/\hbar} + v(\vec{r}) e^{-i\varepsilon t/\hbar} \]

This yields the Bogoliubov (RPA) equations
Scattering of quasiparticles

**Bogoliubov equations:**

\[
(T + V_d)u(r) + V_0 v(r) = (\varepsilon + \mu)u(r) \\
(T + V_d)v(r) + V_0^* u(r) = (-\varepsilon + \mu)v(r)
\]

with the interactions

\[
V_d(r) = V_{\text{ext}}(r) + 2\frac{4\pi a_s N}{m}|\phi(r)|^2 \\
V_0(r) = \frac{4\pi a_s N}{m}\phi(r)^2
\]

This is a two-channel scattering problem – Resonances, Levinson Theorem?
Bogoliubov equations

Need to solve this non-Hermitian eigenvalue problem:

\[
\begin{pmatrix}
A & B \\
-B^* & -A
\end{pmatrix}
\begin{pmatrix}
u(r) \\
v(r)
\end{pmatrix} = \epsilon
\begin{pmatrix}
u(r) \\
v(r)
\end{pmatrix}
\]

with

\[
A = -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}}(r) - \mu + 2g|\psi(r)|^2 \\
B = g\psi(r)^2
\]

where \(\psi(r)\) is a solution of the stationary GP equation

\[
\mu \psi(r) = \left[-\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}}(r) + g|\psi(r)|^2\right] \psi(r)
\]

Properties:

\[
A^\dagger = A
\]

\[
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\begin{pmatrix}
A & B \\
-B^* & -A
\end{pmatrix}^\dagger
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
= \begin{pmatrix}
A & B \\
-B^* & -A
\end{pmatrix}
\]

Eigenvalues come in pairs, either real or imaginary.
Scattering problem

Can we learn about bound states by taking data from scattering experiments?

Potential scattering (single channel): Yes, with Levinson’s theorem (N. Levinson, 1949)
Bogoliubov scattering

Energy

\[ k_1^2 = \varepsilon + \mu \]

particle scattering

trapping-potential

\( \mu \)

condensate

bound collective excitations

Additional solutions of the Bogoliubov equations by symmetry
Levinson’s theorem for BECs

The transmission amplitude $T(k)$ carries a phase $\delta(k)$

$$T(k) = e^{-2i\delta(k)}|T(k)|$$

Levinson’s theorem for BECs:

$$\delta(k = 0) = (n_b - 1/2)\pi$$

the phase shift at zero momentum is related to the number of discrete (localised) collective excitations $n_b$

JB, I Häring, and J-M Rost, PRL 91 (2003) 070403

Thursday, 30 June 2011
Bright Solitons are supported in BECs with

- attractive interactions $(g<0)$
- tight wave-guide geometry: quasi-1D system

A single nonspreading wave packet (soliton) is observed.


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Scattering problem with solitons

How robust are solitons?

Soliton
Bose-Einstein condensate

Will solitons decay by interacting with single thermal particles?
What happens in the scattering process?

Solitons are quantum objects with the coldest temperatures
ideal for atom optics, interferometry,
gravitational and inertial sensors

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Quasiparticle scattering on soliton

\[ R(k) = 0 \quad \quad |T(k)| = 1 \]

Bogoliubov scattering from a bright soliton in the cubic NLS is reflectionless! Solitons are transparent.

\[ \phi(x) = \sqrt{N/(2b)}\text{sech}^2(x/b) \]
\[ u_k = A(k)[\text{sech}(x/b^2) + ikb\text{tanh}(x/b) + (kb)^2 - 1]e^{ikx} \]
\[ v_k = A(k)\text{sech}(x/b^2)e^{ikx} \]
\[ T(k) = \left(\frac{kb + i}{kb - i}\right)^2 \quad |T(k)| = 1 \]

Realistic 3D situation: finite reflection leads to dissipative motion of soliton in thermal cloud.


Thursday, 30 June 2011
Resonant scattering

In two-channel scattering we sometimes find resonances of the Fano / Feshbach type, e.g. in atom-atom scattering.

Can we find similar resonances for quasiparticles scattering on BECs?

Let’s see. Go lattice ...
Lattice with nonlinear defect

Consider a BEC in an optical lattice confined by attractive (or repulsive) interactions that are only present within a single well.

The appropriate version of the GP equation is the discrete nonlinear Schrödinger equation (DNLS)

\[ i \frac{\partial \Psi_n}{\partial t} = -\left( \Psi_{n+1} + \Psi_{n-1} \right) - \gamma |\Psi_n|^{2} \Psi_n \delta_{n,n_c} \]

We can find solutions analytically!

Tsironis, Molina, Hennig PRE 50, 2365 (1994)
Solutions of the defect model

There are two types of solutions:

- **Extended (linear) waves** exist far away from defect
  \[ \Psi_n = \Psi_0 \exp(ikn) \exp(-iE_k t) \]
  
  They form a band with dispersion
  \[ E_k \equiv -2 \cos k, \]

- **Localized solutions** decay exponentially
  \[ \Psi_n(t) = b_n(t) = b_{nc} x^{n-n_c} \exp(-iE_b t) \]

The energies lie outside the linear band

\[ E_b = -\sqrt{4 + g^2} \quad g \equiv \gamma b_{nc}^2 \quad (g > 0) \]
Scattering problem

Approach the scattering problem:

• Expand DNLS around localised solution
  \[ \Psi_n(t) = \phi_n(t) + b_n(t) \]
  \[ \phi_n(t) = u_n e^{-iEt} + v_n^* e^{-i(2E_b-E)t} \]

• Derive discrete Bogoliubov equations

\[ Eu_n = -(u_{n+1} + u_{n-1}) - g(2u_{n_c} + v_{n_c}) \delta_{n,n_c} \]
\[ (2E_b - E)v_n = -(v_{n+1} + v_{n-1}) - g(2v_{n_c} + u_{n_c}) \delta_{n,n_c} \]

• Note that there is a localized solution in the decoupled \( \nu \) channel (\( g u_{n_c} = 0 \)):

  \[ v_n = v_{n_c} w^{n-n_c} \quad w = -g + \sqrt{1 + g^2} \]

With energy within the linear band!

\[ E = E_L \equiv 2(E_b + \sqrt{1 + g^2}) \]

Can we expect a Fano resonance at the local mode energy?

The scattering problem can be solved exactly:

\[ T(k) = \frac{4\sin^2 k}{4\sin^2 k + \left(2g + \frac{g^2}{\sqrt{(E_k - 2E_b)^2 - 4E_b}}\right)^2} \]

At the position of the local mode we find a reflection (Fano) resonance.

The reflection and transmission resonances may be useful for switching atom beams!

Numerical simulations with wavepackets confirm the results of stationary scattering theory.

Topological - non-topological soliton bifurcation
Josephson vortex (fluxon) in superconductor

- **Josephson vortex**: identified by a soliton in the relative phase

- One quantum of magnetic flux

Quantum dynamics of a single vortex

A. Wallraff*, A. Lukashenko, J. Lisenfeld, A. Kemp, M. V. Fistul, Y. Koval & A. V. Ustinov

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letters to nature
Quantum tunneling of vortices

A vortex in a 2D BEC can tunnel between two pinning potentials on observable time scales.


Robust superposition state between vortices can be created in 1D strongly interacting Bose (Tonks-Girardeau) gas.

DW Hallwood, T Ernst, JB, PRA 82, 062623 (2010)
Double ring under rotation

Vortices can enter the tunnel barrier between the two rings.

In analogy to long Josephson junctions we call these “Josephson vortices” or “rotational fluxons”

Competing effects of

External rotation: favours different circulation between rings

Tunnel coupling: favours equal phase across the junction

JB, TJ Haigh, U Zülicke, PRA 80, 011602(R) (2009)
Multiple Josephson Vortex Solutions

When $\langle L_z \rangle_o - \langle L_z \rangle_i > 1$, ground state is lattice of Josephson vortices

e.g. $\langle L_z \rangle_o - \langle L_z \rangle_i \sim 3$

JB, TJ Haigh, U Zülicke, PRA 80, 011602(R) (2009)
Single fluxon in linear geometry

Two linearly coupled GP equations
\[
\begin{align*}
i \partial_t \psi_1 &= \left( -\frac{1}{2} \partial_{xx} + |\psi_1|^2 - \mu \right) \psi_1 - k \psi_2 \\
i \partial_t \psi_2 &= \left( -\frac{1}{2} \partial_{xx} + |\psi_2|^2 - \mu \right) \psi_2 - k \psi_1
\end{align*}
\]

have fluxon solutions for \( 0 < k < 1/3 \)
\[
\psi_{1/2}^\text{fl} = \sqrt{1 + k \tanh(2\sqrt{k}x)} \pm i \sqrt{1 - 3k} \sech(2\sqrt{k}x)
\]

or dark soliton solution for \( k > 1/3 \)
\[
\psi_1^\text{ds} = \psi_2^\text{ds} = \sqrt{1 + k \tanh(\sqrt{1 + k} \ x)}
\]

Kaurov, Kuklov, PRA (2005, 2006)
Qadir, Susanto, Matthews, arXiv (2011)
Fluxon - dark soliton bifurcation

Is the bifurcation point an exceptional point?

- square root branch point at $k = 1/3$
- encircling the branch point in the complex plane interchanges fluxon with anti-fluxon
- Can this be achieved physically?
Thanks!

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The end!