On complexified quantum mechanics and space-time

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Quantum Physics with Non-Hermitian Operators
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Work based on:

- D. C. Brody & E. M. Graefe (2011) "On complexified mechanics and coquaternions" *Journal of Physics* A**44**, 072001.
- D. C. Brody & E. M. Graefe (2011) "Six-dimensional space-time from quaternionic quantum mechanics" (arXiv:1105.3604).
- D. C. Brody & E. M. Graefe (2011) "Coquaternionic quantum dynamics for two-level systems" *Acta Polytechnica* **51** (to appear; arXiv:1105.4038).

Quaternions and coquaternions

Quaternions (Hamilton, 1844)

$$i^2 = j^2 = k^2 = ijk = -1$$

and the cyclic relation

$$ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

The squared norm of $q = q_0 + iq_1 + jq_2 + kq_3$ is

$$|q|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2.$$

Coquaternion (Cockle, 1849)

$$i^2 = -1, \quad j^2 = k^2 = ijk = +1$$

and the skew-cyclic relation

$$ij = -ji = k, \quad jk = -kj = -i, \quad ki = -ik = j.$$

The squared norm of $q = q_0 + iq_1 + jq_2 + kq_3$ is

$$|q|^2 = q_0^2 + q_1^2 - q_2^2 - q_3^2.$$

Complex formulation of real mechanics

Hamilton's equations:

$$\dot{p} = -\frac{\partial H}{\partial x}, \qquad \dot{x} = \frac{\partial H}{\partial p}.$$

Introduce complex phase-space variable

$$z = \tfrac{1}{\sqrt{2}}(x + ip).$$

The canonical equation of motion then reads

$$i\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial H}{\partial \bar{z}},$$

which is just the Schrödinger equation (Dirac, 1927):

$$i\frac{\mathrm{d}|z\rangle}{\mathrm{d}t} = \frac{\partial H}{\partial \langle \bar{z}|}, \qquad H = \frac{\langle \bar{z}|\hat{H}|z\rangle}{\langle \bar{z}|z\rangle}.$$

Complexification

Recall that equations of motions are:

$$\dot{p} = -\frac{\partial H}{\partial x}, \quad \dot{x} = \frac{\partial H}{\partial p}; \qquad i\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial H}{\partial \bar{z}}.$$

Complexification (i)

$$x \to x_0 + ix_1, \quad p \to p_0 + ip_1; \qquad z \to \tfrac{1}{\sqrt{2}}[(x_0 - p_1) + i(x_1 + p_0)] \ ?$$

Complexification (ii)

$$x \to x_0 + jx_1, \quad p \to p_0 + jp_1; \qquad z \to \tfrac{1}{\sqrt{2}}[x_0 + ip_0 + jx_1 + kp_1] \ \checkmark$$

Here, i, j, k can be unit quaternions or coquaternions.

- Quaternions $\Leftrightarrow$ Quaternionic quantum mechanics
- Coquaternions $\Leftrightarrow$ PT-symmetric quantum mechanics

Quaternionic/coquaternionic quantum dynamics

One-parameter unitary group $\hat{U}_t$ is generated by a dynamical equation:

$$|\dot{\Psi}\rangle = -\boldsymbol{i}\hat{H}|\Psi\rangle,$$

where $\hat{H}$ is Hermitian, $\boldsymbol{i}$ is skew-Hermitian unitary, and both commute with $\hat{U}_t$.

We let $\hat{H}$ be given arbitrarily, and restrict $\boldsymbol{i}$ to be the unique unit imaginary (co)quaternion that commute with $\hat{H}$.

Quaternionic spin-$\frac{1}{2}$ particle

A generic 2×2 Hermitian Hamiltonian can be expressed in the form

$$\hat{H} = u_0 \mathbb{1} + \sum_{l=1}^{5} u_l \hat{\sigma}_l,$$

where $\{u_l\} \in \mathbb{R}$, and $\{\hat{\sigma}_l$ are the quaternionic Pauli matrices:

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\hat{\sigma}_4 = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}, \quad \hat{\sigma}_5 = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}.$$

Having specified the Hamiltonian, we must select a unit imaginary quaternion such that the evolution operator $\hat{U}_t = \exp(-\boldsymbol{i}\hat{H}t)$ is unitary.

This is given by

$$\boldsymbol{i} = \frac{1}{\nu}(iu_2 + ju_4 + ku_5),$$

where $\nu = \sqrt{u_2^2 + u_4^2 + u_5^2}$.

Quaternionic spin dynamics

To determine the dynamics we introduce a quaternionic Bloch vector:

$$\sigma_l = \langle \Psi|\hat{\sigma}_l|\Psi\rangle/\langle\Psi|\Psi\rangle, \quad l = 1,\ldots,5.$$

Then for each component we can work out the dynamics by making use of the Schrödinger equation.

After rearrangements we deduce that

$$\begin{aligned}
\tfrac{1}{2}\dot{\sigma}_1 &= \nu\sigma_3 - u_3(u_2\sigma_2 + u_4\sigma_4 + u_5\sigma_5)/\nu \\
\tfrac{1}{2}\dot{\sigma}_2 &= (u_2u_3\sigma_1 - u_1u_2\sigma_3 + u_0u_5\sigma_4 - u_0u_4\sigma_5)/\nu \\
\tfrac{1}{2}\dot{\sigma}_3 &= -\nu\sigma_1 + u_1(u_2\sigma_2 + u_4\sigma_4 + u_5\sigma_5)/\nu \\
\tfrac{1}{2}\dot{\sigma}_4 &= (u_3u_4\sigma_1 - u_0u_5\sigma_2 - u_1u_4\sigma_3 + u_0u_2\sigma_5)/\nu \\
\tfrac{1}{2}\dot{\sigma}_5 &= (u_3u_5\sigma_1 + u_0u_4\sigma_2 - u_1u_5\sigma_3 - u_0u_2\sigma_4)/\nu.
\end{aligned}$$

These evolution equations preserve the normalisation condition:

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2 + \sigma_5^2 = 1,$$

which is the defining equation for the state space $S^4 \subset \mathbb{R}^5$.

$\Rightarrow$ SO(5) symmetry.

Dimensional reduction

As in any physical theory modelled on a higher-dimensional space-time, it is important to identify in which way dimensional reduction occurs.

For this purpose, let us define the three spin variables:

$$\sigma_x = \sigma_1, \quad \sigma_y = \frac{1}{\nu}(u_2\sigma_2 + u_4\sigma_4 + u_5\sigma_5), \quad \sigma_z = \sigma_3.$$

Then a calculation making use of the generalised Bloch dynamics shows that

$$\begin{aligned} \tfrac{1}{2}\dot{\sigma}_x &= \nu\sigma_z - u_3\sigma_y \\ \tfrac{1}{2}\dot{\sigma}_y &= u_3\sigma_x - u_1\sigma_z \\ \tfrac{1}{2}\dot{\sigma}_z &= u_1\sigma_y - \nu\sigma_x. \end{aligned}$$

The reduced spin dynamics is thus confined to the state space

$$\sigma_x^2 + \sigma_y^2 + \sigma_z^2 = r^2,$$

where $r \leq 1$ is time independent.

What about the motion of the 'internal' variables of σ_y: σ_2, σ_4, and σ_5?

The motion of these variables lies on a cylinder in $\mathbb{R}^3$:

$$(u_2\sigma_4 - u_4\sigma_2)^2 + (u_4\sigma_5 - u_5\sigma_4)^2 + (u_5\sigma_2 - u_2\sigma_5)^2 = \nu^2 c^2,$$

where

$$c^2 = 1 - r^2$$

is the squared radius of the cylinder.

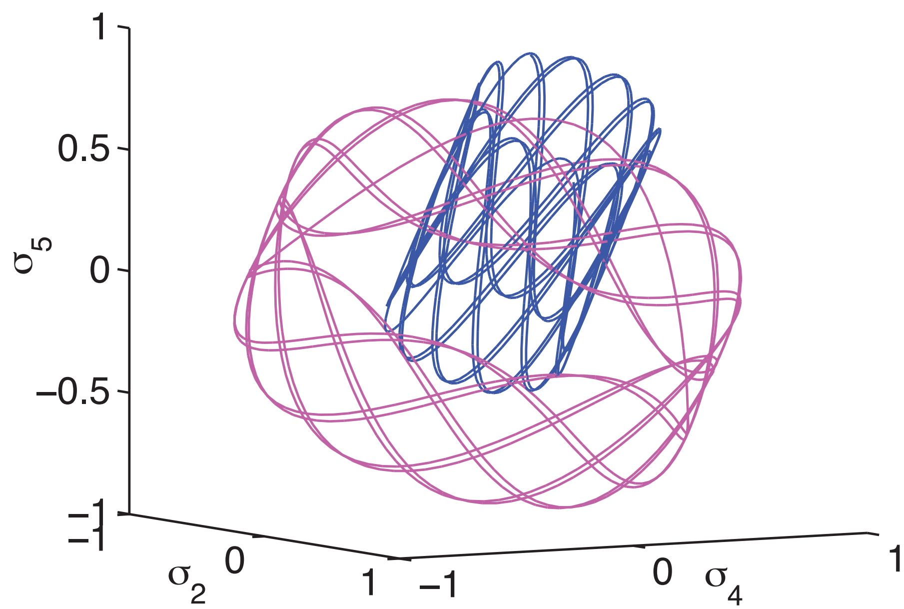

Figure 1: *Dynamical trajectories on the cylindrical subspace.*

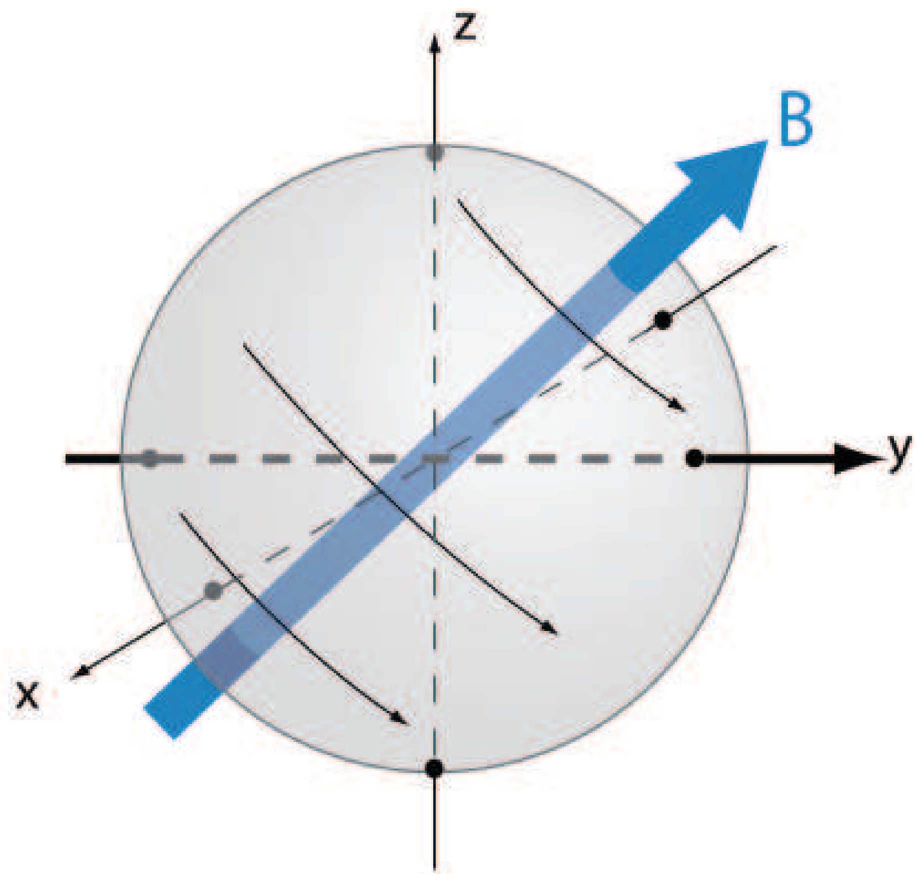

Figure 2: *Bloch dynamics on the reduced state space.*

$\Rightarrow$ Symmetry breaking: $SO(5) \longrightarrow SO(3) \times U(1)$.

Observable effect of extra dimensions

(i) Spin-measurement statistics gives:

$$\sigma_x^2 + \sigma_y^2 + \sigma_z^2 = 1 - c^2$$

(ii) Interference — Peres 1979; Adler 1988; Adler & Anandan 1996

(iii) Superconductor-antiferromagnet — Zhang 1997, 1998, 2000

Coquaternionic spin-$\frac{1}{2}$ particle

A generic 2×2 Hermitian Hamiltonian can be expressed in the form

$$\hat{H} = u_0 \mathbb{1} + \sum_{l=1}^{5} u_l \hat{\sigma}_l,$$

where $\{u_l\} \in \mathbb{R}$, and $\{\hat{\sigma}_l$ are the coquaternionic Pauli matrices:

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$\hat{\sigma}_4 = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}, \quad \hat{\sigma}_5 = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}.$$

The evolution operator $\hat{U}_t = \exp(-\boldsymbol{i}\hat{H}t)$ is unitary if we set

$$\boldsymbol{i} = \frac{1}{\nu}(iu_2 + ju_4 + ku_5),$$

where $\nu = \sqrt{u_2^2 - u_4^2 - u_5^2}$ or $\nu = \sqrt{u_4^2 + u_5^2 - u_2^2}$.

The eigenvalues of H are $E_\pm = u_0 \pm \sqrt{u_1^2 + u_2^2 + u_3^2 - u_4^2 - u_5^2}$.

Polar decomposition of coquaternions

Let $q = q_0 + iq_1 + jq_2 + kq_3$ be a generic coquaternion.

If $\bar{q}q > 0$ and $q_1^2 - q_2^2 - q_3^2 > 0$, then

$$q = |q|\mathrm{e}^{\boldsymbol{i}_q \theta_q} = |q|(\cos\theta_q + \boldsymbol{i}_q \sin\theta_q),$$

where

$$\boldsymbol{i}_q = \frac{iq_1 + jq_2 + kq_3}{\sqrt{q_1^2 - q_2^2 - q_3^2}} \quad \text{and} \quad \theta_q = \tan^{-1}\left(\frac{\sqrt{q_1^2 - q_2^2 - q_3^2}}{q_0}\right).$$

If $\bar{q}q > 0$ but $q_1^2 - q_2^2 - q_3^2 < 0$, then

$$q = |q|\mathrm{e}^{\boldsymbol{i}_q \theta_q} = |q|(\cosh\theta_q + \boldsymbol{i}_q \sinh\theta_q),$$

where

$$\boldsymbol{i}_q = \frac{iq_1 + jq_2 + kq_3}{\sqrt{-q_1^2 + q_2^2 + q_3^2}} \quad \text{and} \quad \theta_q = \tanh^{-1}\left(\frac{\sqrt{-q_1^2 + q_2^2 + q_3^2}}{|q_0|}\right).$$

If $\bar{q}q > 0$ and $q_1^2 - q_2^2 - q_3^2 = 0$, then $q = q_0(1 + \boldsymbol{i}_q)$, where $\boldsymbol{i}_q = (iq_1 + jq_2 + kq_3)/q_0$.

Finally, if $\bar{q}q < 0$, then

$$q = |q|\mathrm{e}^{\boldsymbol{i}_q\theta_q} = |q|(\sinh\theta_q + \boldsymbol{i}_q\cosh\theta_q),$$

where

$$\boldsymbol{i}_q = \frac{iq_1 + jq_2 + kq_3}{\sqrt{-q_1^2 + q_2^2 + q_3^2}} \quad \text{and} \quad \theta_q = \tanh^{-1}\left(\frac{\sqrt{-q_1^2 + q_2^2 + q_3^2}}{q_0}\right).$$

Coquaternionic spin dynamics

In the case of a coquaternionic spin-$\frac{1}{2}$ particle, the generalised Bloch equations are:

$$\begin{aligned}
\tfrac{1}{2}\dot{\sigma}_1 &= \nu\sigma_3 - \frac{u_3}{\nu}(u_2\sigma_2 + u_4\sigma_4 + u_5\sigma_5) \\
\tfrac{1}{2}\dot{\sigma}_2 &= \frac{1}{\nu}(u_2u_3\sigma_1 - u_1u_2\sigma_3 + u_0u_5\sigma_4 - u_0u_4\sigma_5) \\
\tfrac{1}{2}\dot{\sigma}_3 &= -\nu\sigma_1 + \frac{u_1}{\nu}(u_2\sigma_2 + u_4\sigma_4 + u_5\sigma_5) \\
\tfrac{1}{2}\dot{\sigma}_4 &= \frac{1}{\nu}(-u_3u_4\sigma_1 + u_0u_5\sigma_2 + u_1u_4\sigma_3 + u_0u_2\sigma_5) \\
\tfrac{1}{2}\dot{\sigma}_5 &= \frac{1}{\nu}(-u_3u_5\sigma_1 - u_0u_4\sigma_2 + u_1u_5\sigma_3 - u_0u_2\sigma_4),
\end{aligned}$$

These evolution equations preserve the normalisation condition:

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - \sigma_4^2 - \sigma_5^2 = 1,$$

which is the defining equation for the hyperbolic state space.

$\Rightarrow$ SO(3,2) symmetry

Dimensional reduction: Hermitian case

As before, we define the three 'reduced' spin variables

$$\sigma_x = \sigma_1, \quad \sigma_y = \frac{1}{\nu}(u_2\sigma_2 + u_4\sigma_4 + u_5\sigma_5), \quad \sigma_z = \sigma_3.$$

Then a short calculation shows that

$$\begin{aligned} \tfrac{1}{2}\dot{\sigma}_x &= \nu\sigma_z - u_3\sigma_y \\ \tfrac{1}{2}\dot{\sigma}_y &= u_3\sigma_x - u_1\sigma_z \\ \tfrac{1}{2}\dot{\sigma}_z &= u_1\sigma_y - \nu\sigma_x, \end{aligned}$$

when $u_4^2 + u_5^2 < u_2^2$.

This preserves

$$\sigma_x^2 + \sigma_y^2 + \sigma_z^2 = r^2.$$

$\Rightarrow$ Standard unitary dynamics.

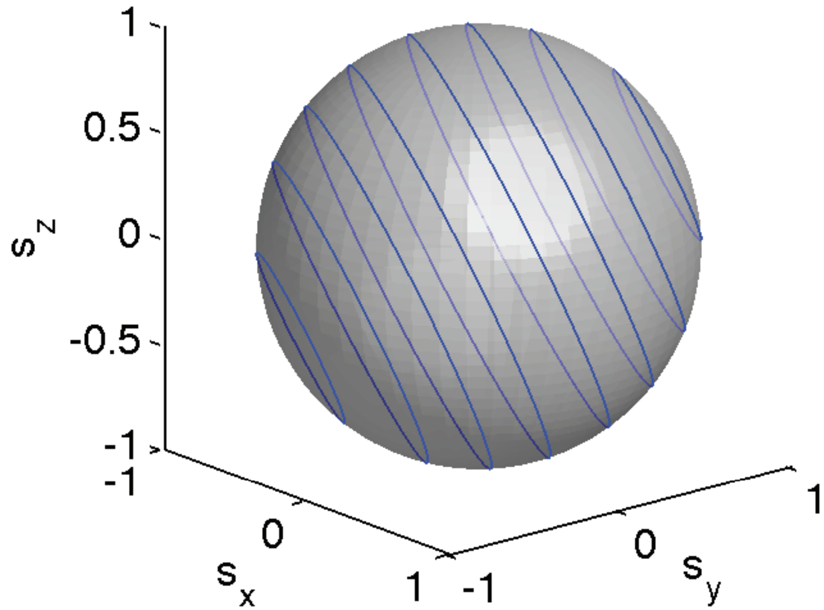

Figure 3: *Reduced Bloch dynamics for a coquaternionic spin-$\frac{1}{2}$ particle.*

Dimensional reduction: non-Hermitian, real energy case

When $u_4^2 + u_5^2 > u_2^2$, we have

$$\begin{aligned} \tfrac{1}{2}\dot{\sigma}_x &= -\nu\sigma_z - u_3\sigma_y \\ \tfrac{1}{2}\dot{\sigma}_y &= -u_3\sigma_x + u_1\sigma_z \\ \tfrac{1}{2}\dot{\sigma}_z &= u_1\sigma_y + \nu\sigma_x. \end{aligned}$$

$\Rightarrow$ Unitary dynamics on a hyperboloid:

$$\sigma_x^2 - \sigma_y^2 + \sigma_z^2 = r^2.$$

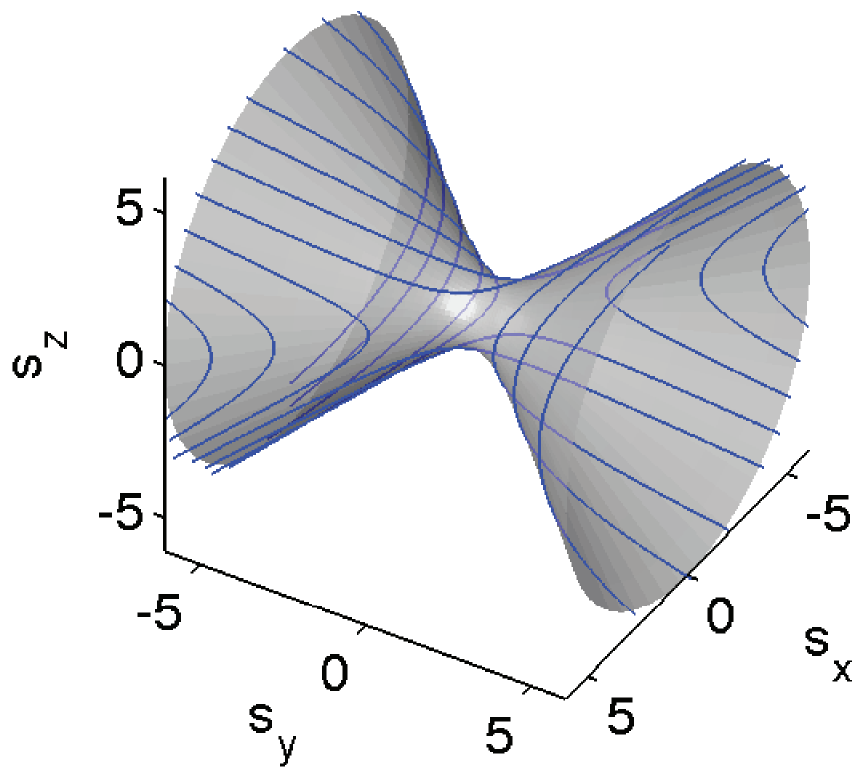

Figure 4: *Reduced dynamics for a coquaternionic spin-$\frac{1}{2}$ particle with unbroken symmetry.*

Dimensional reduction: non-Hermitian, real energy case

When $u_4^2 + u_5^2 > u_2^2$, and $u_1^2 + u_2^2 + u_3^2 > u_4^2 + u_5^2$ so that energy eigenvalues form complex conjugate pairs, the orbits close on a hyperboloid.

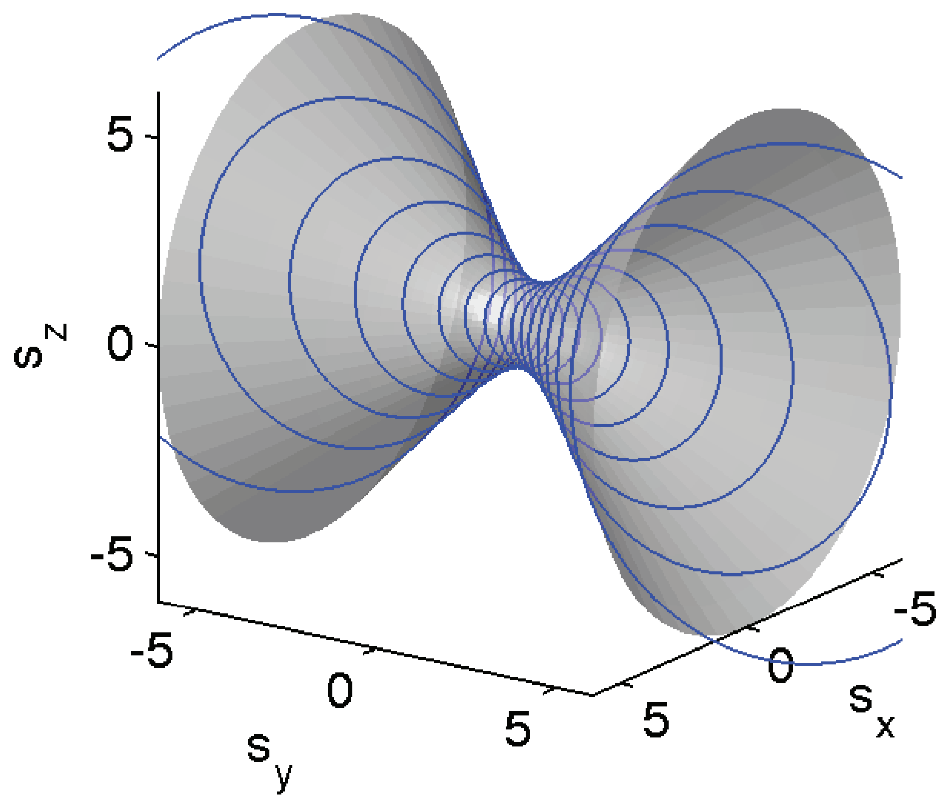

Figure 5: *Reduced dynamics for a coquaternionic spin-$\frac{1}{2}$ particle with broken symmetry.*

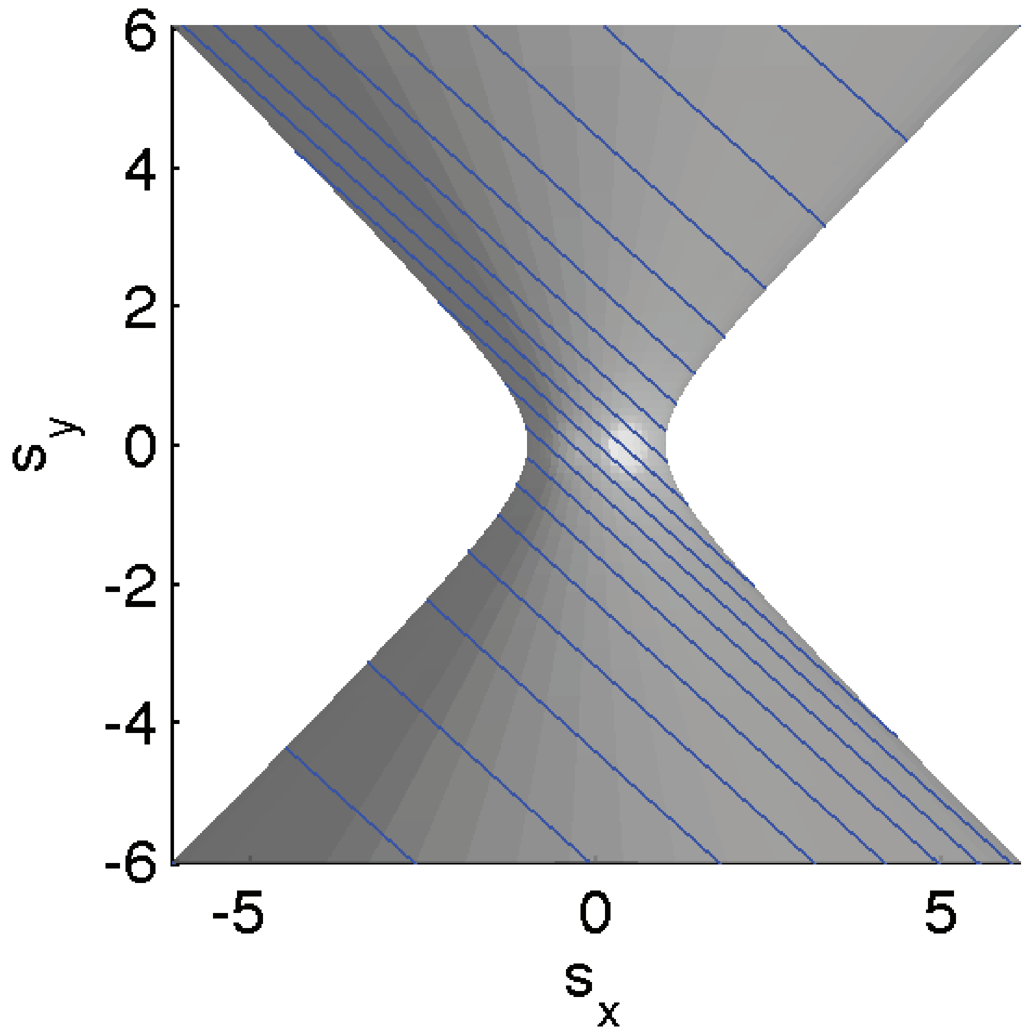

Figure 6: *Reduced dynamics for a coquaternionic spin-$\frac{1}{2}$ particle with broken symmetry.*

Closing remarks

Symplectization, Complexification and Mathematical Trinities

V. I. Arnold (1997)

"Maybe there is some complexified version of the quantum Hall effect, the three dimensional transversal being replaced by a five dimensional one. It would have been easy to predict the quantum Hall effect and the Berry phase theory simply by complexifying the theory of monodromy of quadratic forms from the Modes and Quasimodes. This opportunity was lost. We may also miss more opportunities not studying the quaternionic version of the modes and quasimodes theory."

coquaternionic

Open questions

- Relation to the $\mathcal{C}$ operator
- Mapping from the hyperbolic state space to the spherical state space
- Infinite-dimensional case and boundary conditions