Convergent Quantum Normal Forms, $\mathcal{PT}$-symmetry and reality of the spectrum

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1. Introduction and statement of the results

We address the problem **to conjugate** $\mathcal{PT}$ symmetric quantum operators with **selfadjoint** operators through a similarity transformation with the techniques of the Quantum Normal Form (**QNF**). We provide a class of operators for which the procedure works.

More precisely:

1) We prove the reality of the spectrum of $\mathcal{PT}$-symmetric non s.a. operators;

2) We obtain an exact quantization formula for the eigenvalues;

3) We determine a similarity transformation that maps the $\mathcal{PT}$-symmetric operator into a selfadjoint one;

4) We construct the QNF which generates the Classical Normal Form (**CNF**) for $\hbar = 0$.

(Recent results obtained with S. Graffi)

Step 1

Start with a $\mathcal{PT}$-symmetric classical Hamiltonian family, expressed in action-angle variables:

$$\mathcal{H}_\varepsilon(\xi, x) = \mathcal{L}_\omega(\xi) + \varepsilon \mathcal{V}_\omega(\xi, x)$$

in $\mathbf{R}^l \times \mathbf{T}^l, \varepsilon \in \mathbf{R}$, $\quad \mathcal{L}_\omega(\xi) := \langle \omega, \xi \rangle$;

$\mathcal{P}$: $\quad x \to -x$; $\qquad \mathcal{T}$: complex conjugation

Weyl quantization formula (**WQF**) yields the $\mathcal{PT}$-symmetric, non s.a. operator in $L^2(\mathbf{T}^l)$:

$$H(\varepsilon) = i\hbar\langle \omega, \nabla \rangle + \varepsilon V = L(\omega, \hbar) + \varepsilon V$$

complex holomorphic perturbation of the linear diophantine flow over $\mathbf{T}^l$, $\mathcal{PT}$-symmetric.

Step 2

Construct the Operator Quantum Normal Form (**O-QNF**) which diagonalizes $H(\varepsilon)$ by means of a similarity transformation:

$$e^{iW(\varepsilon)/\hbar}H(\varepsilon)e^{-iW(\varepsilon)/\hbar} = S(\varepsilon)$$

$$= L(\omega,\hbar) + \sum_{k=0}^{\infty} \varepsilon^k B_k(\hbar)$$

where $\quad [B_k, L] = 0, \ \forall k$

Step 3

Look for $W(\varepsilon)$ such that $S(\varepsilon)$ is selfadjoint, thus providing a real spectrum.

To this end:

- Construct the QNF for the symbols (**S-QNF**) to determine $\Sigma(\varepsilon)$, symbol of $S(\varepsilon)$:

$$\Sigma(\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \mathcal{B}_k(\xi,\hbar) \ , \quad \mathcal{B}_0 = \mathcal{L}_\omega(\xi)$$

- Pass from the symbols to the operators through the WQF; in this case, if applied to $\mathcal{B}_k(\xi,\hbar)$, symbol of B_k, it provides an exact quantization formula for the eigenvalues of $H(\varepsilon)$.

More precisely:

(1) The **series converges** (Graffi-Paul 2011); therefore the eigenvalues are given by:

$$\lambda_n(\hbar,\varepsilon) = \langle\omega,n\rangle\hbar + \sum_{k=1}^{\infty} \mathcal{B}_k(n\hbar,\hbar)\varepsilon^k, \quad n \in \mathbf{Z}^l$$

(2) We prove that **each** $\mathcal{B}_k(\xi,\hbar)$ **is real**, thus each operator $B_k = B_k^*$ is **s.a.** $\Rightarrow S(\varepsilon) = S(\varepsilon)^*$.

Remark

The unperturbed spectrum is pure point but dense, therefore the standard perturbation theory cannot be applied here; the approach through the Normal Form is necessary.

Therefore we provide an explicit construction of the similarity transformation mapping a $\mathcal{PT}$ symmetric operator into a s.a. operator.

Moreover:

(3) $\mathcal{B}_k(\xi,0) \equiv \mathcal{B}_k(\xi)$ is the k-th coefficient of the CNF of the classical Hamiltonian $\mathcal{H}_\varepsilon(\xi,x)$:

$$\lim_{\substack{\hbar\to 0\\ n\hbar\to\xi}} \mathcal{B}_k(n\hbar,\hbar) = \mathcal{B}_k(\xi,0) = \mathcal{B}^c_k(\xi)$$

integrable system mapped into a real Hamiltonian.

An application to classical mechanics:

$\mathcal{PT}$-symmetric, non-holomorphic perturbations of non-resonant harmonic oscillators

Consider the inverse transformation into action-angle variables

$$\mathcal{C}(\xi, x) = (\eta, y) := \begin{cases} \eta_i = -\sqrt{\xi_i}\,\text{sin}\, x_i, \\ \\ y_i = \sqrt{\xi_i}\,\text{cos}\, x_i, \end{cases} \qquad i = 1, \ldots, l$$

It is defined only on $\mathbf{R}^l_+ \times \mathbf{T}^l$ and does not preserve the regularity at the origin. On the other hand, $\mathcal{C}$ is an analytic, canonical map between $\mathbf{R}^l_+ \times \mathbf{T}^l$ and $\mathbf{R}^{2l} \setminus \{0, 0\}$.

Then

$$(\mathcal{H}_\varepsilon \circ \mathcal{C}^{-1})(\eta, y) = \sum_{s=1}^{l} \omega_s(\eta_s^2 + y_s^2) + \varepsilon(\mathcal{V} \circ \mathcal{C}^{-1})(\eta, y)$$

$$:= \mathcal{P}_0(\eta, y) + \varepsilon \mathcal{P}_1(\eta, y)$$

where for $(\eta, y) \in \mathbf{R}^{2l} \setminus \{0, 0\}$

$$\mathcal{P}_1(\eta, y) = (\mathcal{V} \circ \mathcal{C}^{-1})(\eta, y) = \mathcal{P}_{1,R}(\eta, y) + \mathcal{P}_{1,I}(\eta, y),$$

$$\mathcal{P}_{1,R}(\eta,y) = \frac{1}{2}\sum_{k\in\mathbf{Z}^l}(\mathrm{Re}\,\mathcal{V}_k\circ\mathcal{C}^{-1})(\eta,y)\prod_{s=1}^{l}\left(\frac{\eta_s - iy_s}{\sqrt{\eta_s^2+y_s^2}}\right)^{k_s}$$

$$\mathcal{P}_{1,I}(\eta,y) = \frac{1}{2}\sum_{k\in\mathbf{Z}^l}(\mathrm{Im}\mathcal{V}_k\circ\mathcal{C}^{-1})(\eta,y)\prod_{s=1}^{l}\left(\frac{\eta_s - iy_s}{\sqrt{\eta_s^2+y_s^2}}\right)^{k_s}$$

Under suitable assumptions on ω and $\mathcal{V}$ (see below) we obtain

Proposition 1

The Birkhoff normal form of $\mathcal{H}_\varepsilon$ *is* <u>*real*</u> *and uniformly convergent on any compact of* $\mathbf{R}^{2l}\setminus\{0,0\}$ *if* $|\varepsilon| < \varepsilon_0$. *Hence the system is integrable.*

2. Reminder on Weyl's quantization formula

Let us sum up the canonical (Weyl) quantization procedure for functions (classical observables) defined on the phase space $\mathbf{R}^l \times \mathbf{T}^l$.

Let $\mathcal{A}(\xi,x,\hbar) : \mathbf{R}^l \times \mathbf{T}^l \times [0,1] \to \mathbf{C}$ be a family of smooth phase-space functions indexed by $\hbar$ written under its Fourier representation

$$\mathcal{A}(\xi,x,\hbar) = \int_{\mathbf{R}^l} \sum_{q\in\mathbf{Z}^l} \widehat{\mathcal{A}}_q(p;\hbar) e^{i(\langle p,\xi\rangle + \langle q,x\rangle)}\, dp$$

Then the (Weyl) quantization of $\mathcal{A}(\xi,x;\hbar)$ is the operator acting on $L^2(\mathbf{T}^l)$, defined by:

$$(A(\hbar)f)(x) :=$$

$$= \int_{\mathbf{R}^l} \sum_{q\in\mathbf{Z}^l} \widehat{\mathcal{A}}_q(p;\hbar) e^{i(\langle q,x\rangle + \langle p,q\rangle\hbar/2)} f(x+p\hbar)\, dp, \quad (12)$$

$\forall f \in L^2(\mathbf{T}^l)$.

Remark 1

If $\mathcal{A}(\xi,x;\hbar) = \mathcal{A}_\omega(\xi,x;\hbar) = \mathcal{A}(\langle\omega,\xi\rangle,x;\hbar)$

(12) clearly becomes:

$$(A(\hbar)f)(x)$$

$$= \int_{\mathbf{R}} \sum_{q\in\mathbf{Z}^l} \widehat{\mathcal{A}}_q(p;\hbar) e^{i(\langle q,x\rangle + p\langle\omega,q\rangle\hbar/2)} f(x+p\hbar\omega)\,dp$$

Remark 2

If $\mathcal{A}$ does not depend on ξ, $\mathcal{A}(\xi,x,\hbar)=\mathcal{A}(x,\hbar)$, (12) reduces to the standard multiplicative action:

$$(A(\hbar)f)(x)$$

$$= \int_{\mathbf{R}^l} \sum_{q\in\mathbf{Z}^l} \mathcal{A}_q(\hbar)\delta(p) e^{i(\langle q,x\rangle + \langle p,q\rangle\hbar/2)} f(x+\hbar p)\,dp$$

$$= \sum_{q\in\mathbf{Z}^l} \mathcal{A}_q(\hbar) e^{i\langle q,x\rangle} f(x) = \mathcal{A}(x,\hbar) f(x)$$

Remark 3

If $\mathcal{A}$ does not depend on x, then $\widehat{\mathcal{A}}_q = 0, q \neq 0$; thus $\widehat{\mathcal{A}}_0 = \widehat{\mathcal{A}}(p,\hbar)$ and the standard (pseudo) differential action is recovered:

$$(A(\hbar)f)(x) = \int_{\mathbf{R}^l} \widehat{\mathcal{A}}(p,\hbar) f(x+\hbar p)\, dp$$

$$= \int_{\mathbf{R}^l} \sum_{q\in\mathbf{Z}^l} \widehat{\mathcal{A}}(p,\hbar) f_q e^{i\langle q, x+\hbar p\rangle}\, dp = \sum_{q\in\mathbf{Z}^l} f_q \mathcal{A}(q\hbar,\hbar) e^{i\langle q,x\rangle}$$

$$= (\mathcal{A}(-i\hbar\nabla_x,\hbar)f)(x),$$

whence the formula for the spectrum of A:

$\lambda_n(\hbar) = \langle e_n, Ae_n\rangle = \mathcal{A}(n\hbar,\hbar)$

Proposition 2

If $\exists \rho \geq 0$ *such that*

$$\|\mathcal{A}\|_\rho := \sup_{\hbar\in[0,1]} \sum_{q\in\mathbf{Z}^l} e^{\rho|q|} \int_{\mathbf{R}^l} e^{\rho|p|} |\widehat{\mathcal{A}}_q(p,\hbar)|\, dp < +\infty,$$

then $A(\hbar)$ *is a uniformly bounded operator in* $L^2(\mathbf{T}^l)$*, because:*

$$\|A(\hbar)\|_{L^2\to L^2} \leq \|\mathcal{A}\|_\rho.$$

3. Assumptions on $\mathcal{H}_\varepsilon(\xi,x)=\mathcal{L}_\omega(\xi)+\varepsilon\mathcal{V}_\omega(\xi,x)$

$$L(\omega,\hbar)\psi = i\hbar\langle\omega,\nabla\rangle\psi$$

$$= -i\hbar[\omega_1\frac{\partial\psi}{\partial x_1}+\ldots+\omega_l\frac{\partial\psi}{\partial x_l}], \qquad \forall\psi\in\mathcal{H}^1(\mathbf{T}^l)$$

ω: diophantine frequencies, i.e. $\exists\gamma>0,\ \tau>l-1$:

$$|\langle\omega,q\rangle|^{-1}\le\gamma|q|^\tau, \qquad q\in\mathbf{Z}^l,\ q\neq 0.$$

V: Weyl quantization of $\mathcal{V}_\omega(\xi,x):\mathbf{R}^l\times\mathbf{T}^l\to\mathbf{C}$

s.t.

$\mathcal{V}_\omega(\xi,x):=\mathcal{V}(\langle\omega,\xi\rangle,x);\ \mathcal{V}:\mathbf{R}\times\mathbf{T}^l\to\mathbf{C}$

$\mathcal{V}(t,x)=\sum_{q\in\mathbf{Z}^l}\mathcal{V}_q(t)e^{i\langle q,x\rangle}$

Space Fourier transform of $\mathcal{V}_q(t)$:

$\widehat{\mathcal{V}}_q(p):=\frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}}\mathcal{V}_q(t)e^{-ipt}\,dt, \qquad p\in\mathbf{R}.$

Then the Weyl quantization of $\mathcal{V}_\omega(\xi,x)$ is:

$$V_\omega f(x)=\int_{\mathbf{R}}\sum_{q\in\mathbf{Z}^l}\widehat{\mathcal{V}}_q(p)e^{i\langle q,x\rangle+\hbar p\langle\omega,q\rangle/2}f(x+\hbar p\omega)\,dp.$$

$\mathcal{PT}$-symmetry assumptions

On the classical potential $\mathcal{V}_\omega(\xi,x)$:

$$\mathcal{V}_{\omega,-q}(\xi) = -\mathcal{V}_{\omega,q}(\xi) \in \mathbf{R}$$

$$\mathcal{V}_{\omega,q}(-\xi) = \mathcal{V}_{\omega,q}(\xi), \quad \forall (\xi,q) \in \mathbf{R}^l \times \mathbf{T}^l$$

Then: $\widehat{\mathcal{V}}_q(-p) = \widehat{\mathcal{V}}_q(p) \in \mathbf{R}, \quad \forall q$ and

$$(\mathcal{PT})\mathcal{V}_\omega(\xi,x) = (\mathcal{PT})(\sum_{q\in\mathbf{Z}^l} \mathcal{V}_{\omega,q}(\xi)e^{i\langle q,x\rangle}) = \mathcal{V}_\omega(\xi,x)$$

Then $V := V_\omega$ is a $\mathcal{PT}$-symmetric operator in $L^2(\mathbf{T}^l)$:

$$(\mathcal{PT})(Vf)(x)$$

$$= \int_{\mathbf{R}} \sum_{q\in\mathbf{Z}^l} \widehat{\mathcal{V}}_q(p) e^{i\langle q,x\rangle - i\hbar p\langle\omega,q\rangle/2} \overline{f}(-x+\hbar p\omega)\,dp$$

$$= \int_{\mathbf{R}} \sum_{q\in\mathbf{Z}^l} \widehat{\mathcal{V}}_q(p) e^{i(\langle q,x\rangle + \hbar p\langle\omega,q\rangle/2)} \overline{f}(-x-\hbar p\omega)\,dp$$

$$= \int_{\mathbf{R}} \sum_{q\in\mathbf{Z}^l} \widehat{\mathcal{V}}_q(p) e^{i(\langle q,x\rangle + \hbar p\langle\omega,q\rangle/2)} (\mathcal{PT}f)(x+\hbar p\omega)\,dp$$

$$= V(\mathcal{PT}f)(x).$$

Boundedness assumption

(to ensure the uniform convergence of the QNF):

$\exists \rho > 1 + 16\gamma\tau^{\tau}$ s.t.

$$\|\mathcal{V}_\omega\|_\rho := \sum_{q \in \mathbf{Z}^l} e^{\rho|q|} \int_{\mathbf{R}} e^{\rho|p|} |\widehat{\mathcal{V}}_q(p)| \, dp < +\infty,$$

which implies the boundedness of the operator V, since

$$\|V\|_{L^2 \to L^2} \leq \|\mathcal{V}_\omega\|_\rho.$$

Then the symbol of the operator

$$H(\varepsilon) = i\hbar\langle \omega, \nabla \rangle + \varepsilon V$$

is the Hamiltonian family:

$\mathcal{H}_\varepsilon(\xi, x) = \langle \omega, \xi \rangle + \varepsilon \mathcal{V}_\omega(\xi, x) = \mathcal{L}_\omega(\xi) + \varepsilon \mathcal{V}_\omega(\xi, x)$

Moreover: $D(H(\varepsilon)) = \mathcal{H}^1(\mathbf{T}^l)$ and denote

$\sigma(H(\varepsilon)) = \{\lambda_n(\hbar, \varepsilon) : \; n \in \mathbf{Z}^l\}$

the spectrum of $H(\varepsilon)$.

Main Result

Under the above assumptions, $\exists \varepsilon_0 > 0$ *independent of* $\hbar$*: for* $|\varepsilon| < \varepsilon_0$ *the spectrum of* $H(\varepsilon)$ *is given by the exact quantization formula:*

$$\lambda_n(\hbar, \varepsilon) = \langle \omega, n \rangle \hbar + \mathcal{B}(n\hbar, \hbar; \varepsilon), \quad n \in \mathbf{Z}^l$$

$$\mathcal{B}(n\hbar, \hbar; \varepsilon) := \sum_{k=1}^{\infty} \mathcal{B}_k(n\hbar, \hbar)\varepsilon^k \qquad (1)$$

where

1) $\mathcal{B}_k(\xi, \hbar) \in C^\infty(\mathbf{R}^l \times [0,1]; \mathbf{R})$, $k = 1, 2, \ldots$

2) *the series (1) converges uniformly w.r.t.* $(\xi, \hbar) \in \mathbf{R}^l \times [0,1]$*;*

3) $\mathcal{B}_k(\xi, 0)$ *is the* $k-$*th coefficient of the CNF for* $\mathcal{H}_\varepsilon(\xi, x)$*;*

4) $\mathcal{B}_k(n\hbar, \hbar)$ *is obtained from the WQF applied to* $\mathcal{B}_k(\xi, \hbar)$*, which is the symbol of the operator* B_k*, the term of order* k *of the QNF.*

Corollary

The operator $S(\varepsilon)$, similar to $H(\varepsilon)$, is selfadjoint.

The Quantum Normal Form: the formal construction

(We follow Sjöstrand (1991) and Bambusi-Graffi-Paul (1999))

Given $H(\varepsilon) = L(\omega, \hbar) + \varepsilon V$ in $L^2(\mathbf{T}^l)$, look for a similarity transformation, in general <u>non unitary</u> $(W(\varepsilon) \neq W(\varepsilon)^*)$

$U(\omega, \varepsilon, \hbar) = e^{iW(\varepsilon)/\hbar} : L^2(\mathbf{T}^l) \leftrightarrow L^2(\mathbf{T}^l)$, s.t.

$$S(\varepsilon) := UH(\varepsilon)U^{-1} = L(\omega, \hbar) + \varepsilon B_1 + \varepsilon^2 B_2 + \ldots$$

$$= L(\omega, \hbar) + \sum_{k=1}^{\infty} B_k \varepsilon^k \qquad (2)$$

under the requirement:

$$[B_k, L] = 0, \forall k.$$

Recall the formal commutator expansion

$$S(\varepsilon) = e^{iW(\varepsilon)/\hbar} H(\varepsilon) e^{-iW(\varepsilon)/\hbar} = \sum_{k=0}^{\infty} H_k$$

$$H_0 := H(\varepsilon), \quad H_k := \frac{[W(\varepsilon), H_{k-1}]}{i\hbar k}, \qquad k \geq 1$$

Look for $W(\varepsilon)$ in the form of a power series expansion in ε: $\quad W(\varepsilon) = \varepsilon W_1 + \varepsilon^2 W_2 + \ldots.$

Then (2) becomes:

$$S(\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k B_k$$

where

$$B_0 = L(\omega, \hbar); \quad B_k := \frac{[W_k, L]}{i\hbar} + V_k\,, \qquad k \geq 1,$$

$$V_1 \equiv V$$

$$V_k = \sum_{r=2}^{k} \frac{1}{r!} \sum_{\substack{j_1+\ldots+j_r=k \\ j_s \geq 1}} \frac{[W_{j_1}, [W_{j_2}, \ldots, [W_{j_r}, L] \ldots]}{(i\hbar)^r}$$

$$+ \sum_{r=1}^{k-1} \frac{1}{r!} \sum_{\substack{j_1+\ldots+j_r=k-1 \\ j_s \geq 1}} \frac{[W_{j_1}, [W_{j_2}, \ldots, [W_{j_r}, V] \ldots]}{(i\hbar)^r}.$$

V_k depends on $W_1, \ldots, W_{k-1}$ (not on W_k!), thus we get the recursive homological equations:

$$\frac{[W_k, L]}{i\hbar} + V_k = B_k, \qquad [L, B_k] = 0 \qquad (3)$$

To solve (3) for $S(\varepsilon), B_k, W_k$, look for their symbols and then apply the WQF. Recall that the symbol of $[F, G]/i\hbar$ is the **Moyal bracket** $\{\mathcal{F}, \mathcal{G}\}_M$ of the symbols $\mathcal{F}$ of F and $\mathcal{G}$ of G, where:

$$\{\mathcal{F}, \mathcal{G}\}_M \sim \sum_{s=0}^{\infty} (-1)^s \hbar^s \sum_{r=0}^{2s+1} \frac{(-1)^r}{r!(2s+1-r)!}$$

$$\times \left(\frac{\partial^{2s+1} \mathcal{F}}{\partial \xi^{2s+1-r} \partial x^r} \right) \left(\frac{\partial^{2s+1} \mathcal{G}}{\partial x^{2s+1-r} \partial \xi^r} \right)$$

$$= \{\mathcal{F}, \mathcal{G}\} + O(\hbar^2).$$

The above equations become, once written for the symbols: $\Sigma(\varepsilon) = \sum_{k=0}^{\infty} \mathcal{H}_k$

$$\mathcal{H}_0 := \mathcal{L}_\omega + \varepsilon \mathcal{V}, \quad \mathcal{H}_k := \frac{\{\mathcal{W}(\varepsilon), \mathcal{H}_{k-1}\}_M}{k}, \ k \geq 1$$

where $\mathcal{W}(\varepsilon) = \varepsilon\mathcal{W}_1 + \varepsilon^2\mathcal{W}_2 + \dots$,
$\Sigma(\varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k \mathcal{B}_k$ and:

$$\mathcal{B}_0 = \mathcal{L}_\omega = \langle \omega, \xi \rangle; \quad \mathcal{B}_k = \{\mathcal{W}_k, \mathcal{L}\}_M + \mathcal{V}_k, \ k > 1,$$

$$\mathcal{V}_1 \equiv \mathcal{V}$$

$$\mathcal{V}_k = \sum_{r=2}^{k} \frac{1}{r!} \sum_{\substack{j_1+\dots+j_r=k \\ j_s \geq 1}} \{\mathcal{W}_{j_1}, \{\mathcal{W}_{j_2}, \dots, \{\mathcal{W}_{j_r}, \mathcal{L}\}_M \dots\}_M$$

$$+ \sum_{r=1}^{k-1} \frac{1}{r!} \sum_{\substack{j_1+\dots+j_r=k-1 \\ j_s \geq 1}} \{\mathcal{W}_{j_1}, \{\mathcal{W}_{j_2}, \dots, \{\mathcal{W}_{j_r}, \mathcal{V}\}_M \dots\}_M,$$

$$k > 1 \tag{4}$$

Therefore the symbols $\mathcal{W}_k$ and $\mathcal{B}_k$ of W_k and B_k can be recursively found solving the **homological equation**:

$$\{\mathcal{W}_k, \mathcal{L}\}_M + \mathcal{V}_k = \mathcal{B}_k, \qquad k = 1, \dots \tag{5}$$

under the condition: $\{\mathcal{L}, \mathcal{B}_k\}_M = 0. \qquad (6)$

Here:

$\mathcal{W}_k = \mathcal{W}_k(\xi, x; \hbar)$, $\mathcal{V}_k = \mathcal{V}_k(\xi, x; \hbar)$, $\mathcal{B}_k = \mathcal{B}_k(\xi, x; \hbar)$.

Notice that (6) is satisfied if $\mathcal{B}_k = \mathcal{B}_k(\xi; \hbar)$ does not depend on x.

Since $\mathcal{L} = \mathcal{L}(\xi) = \langle \omega, \xi \rangle$ is linear in ξ:

$\{\mathcal{W}_k, \mathcal{L}\}_M = \{\mathcal{W}_k, \mathcal{L}\} = -\langle \nabla_x \mathcal{W}_k, \omega \rangle$,

(5) becomes

$$-\langle \nabla_x \mathcal{W}_k(\xi, x), \omega \rangle + \mathcal{V}_k(\xi, x) = \mathcal{B}_k(\xi) \qquad (7)$$

In terms of the Fourier coefficients of

$\mathcal{W}_k(\xi, x) = \sum_{q \in \mathbf{Z}^l} \mathcal{W}_{k,q}(\xi) e^{i\langle q, x \rangle}$ and

$\mathcal{V}_k(\xi, x) = \sum_{q \in \mathbf{Z}^l} \mathcal{V}_{k,q}(\xi) e^{i\langle q, x \rangle}$ (7) becomes:

$$-i \sum_{q \neq 0} \langle k, \omega \rangle \mathcal{W}_{k,q}(\xi) e^{i\langle q, x \rangle} + \sum_{q \in \mathbf{Z}^l} \mathcal{V}_{k,q}(\xi) e^{i\langle q, x \rangle} = \mathcal{B}_k(\xi)$$

whence

$$\mathcal{B}_k(\xi) = \mathcal{V}_{k,0}(\xi), \qquad \mathcal{W}_{k,q}(\xi) = \frac{\mathcal{V}_{k,q}(\xi)}{i\langle q, \omega \rangle}, \quad \forall k \neq 0.$$

For $k = 1$ we have $(\mathcal{V}_1 \equiv \mathcal{V} = \mathcal{V}_\omega)$

$\mathcal{B}_1(\xi) = \mathcal{V}_{\omega,0}(\xi) \in \mathbf{R}$ and

$$\mathcal{W}_{1,q}(\xi) = \frac{\mathcal{V}_{\omega,q}(\xi)}{i\langle q, \omega\rangle} \in i\mathbf{R},\ q \neq 0 \text{ (p. imaginary).}$$

We can choose $\mathcal{W}_{1,0} = 0$

Assume:

$(\mathbf{A_1})$ $\quad \mathcal{V}_{j,q}(\xi) \in \mathbf{R}, \quad \forall j = 1, \ldots, k-1,\ \forall q \in \mathbf{Z}^l;$

$$(\Rightarrow \mathcal{W}_{j,q}(\xi) = \tfrac{\mathcal{V}_{j,q}(\xi)}{i\langle q,\omega\rangle} \in i\mathbf{R} \text{ and } \mathcal{B}_j(\xi) = \mathcal{V}_{j,0} \in \mathbf{R})$$

$(\mathbf{A_2})$ We can choose $\mathcal{W}_{j,0} = 0$, $\forall j = 1, \ldots, k-1$.

Then:

$(\mathbf{R_1})$ $\quad \mathcal{V}_{k,q}(\xi) \in \mathbf{R},\ \forall q \in \mathbf{Z}^l;$

$$(\Rightarrow \mathcal{W}_{k,q}(\xi) = \tfrac{\mathcal{V}_{k,q}(\xi)}{i\langle q,\omega\rangle} \in i\mathbf{R} \text{ and } \mathcal{B}_k(\xi) = \mathcal{V}_{k,0} \in \mathbf{R})$$

$(\mathbf{R_2})$ We can choose $\mathcal{W}_{k,0} = 0$.

Sketch of the proof

(A) Let $f(\xi, x) = \sum_{q \in \mathbf{Z}^l} f_q(\xi) e^{i\langle q, x \rangle}$ and $g(\xi, x) = \sum_{q \in \mathbf{Z}^l} g_q(\xi) e^{i\langle q, x \rangle}$ have real Fourier coefficients: $f_q(\xi), g_q(\xi) \in \mathbf{R}$, $\forall q \in \mathbf{Z}^l$.

Then, $\{f, g\}_M$ has purely imaginary Fourier coefficients:

$$\{f, g\}_M \sim \sum_{s=0}^{\infty} (-1)^s \hbar^s \sum_{r=0}^{2s+1} \frac{(-1)^r}{r!(2s+1-r)!}$$

$$\times \left(\frac{\partial^{2s+1} f}{\partial \xi^{2s+1-r} \partial x^r} \right) \left(\frac{\partial^{2s+1} g}{\partial x^{2s+1-r} \partial \xi^r} \right) \qquad (8)$$

Each derivative w.r.t. x generates a factor i in the Fourier coefficients; so in each summand in (8) we can factor $(i)^{2s+1} = (-1)^s i$.

Thus, in

$$\mathcal{V}_k = \sum_{r=2}^{k} \frac{1}{r!} \sum_{\substack{j_1+\ldots+j_r=k \\ j_s \geq 1}} \{\mathcal{W}_{j_1}, \{\mathcal{W}_{j_2}, \ldots, \{\mathcal{W}_{j_r}, \mathcal{L}\}_M \ldots\}_M$$

$$+ \sum_{r=1}^{k-1} \frac{1}{r!} \sum_{\substack{j_1+\ldots+j_r=k-1 \\ j_s \geq 1}} \{\mathcal{W}_{j_1}, \{\mathcal{W}_{j_2}, \ldots, \{\mathcal{W}_{j_r}, \mathcal{V}\}_M \ldots\}_M,$$

we first factor $(i)^r$ from the coefficients of each $\mathcal{W}_{j_s}$, then each Moyal bracket generates another factor i. So the coefficients of $\mathcal{V}_k$ can be written as the product of a real term $a_{k,q}(\xi)$ times $(i)^{2r}$:

$$\mathcal{V}_{k,q}(\xi) = (i)^{2r} a_{k,q}(\xi) = (-1)^r a_{k,q}(\xi) \in \mathbf{R}$$

(B) The uniform convergence in $(\xi, \hbar)$ of the S-QNF ensures that also the O-QNF converges and, since $\Sigma(\varepsilon)$ is real, that $S(\varepsilon)$ is selfadjoint and the exact quantization of the eigenvalues.

(C) Moreover, since $\mathcal{V}_\omega(\xi, x)$ is odd in x:

$\mathcal{V}_\omega(\xi, -x) = -\mathcal{V}_\omega(\xi, x)$, we get $\mathcal{B}_{2k+1} = 0, \forall k$;

thus,

$$\Sigma(\varepsilon) = \mathcal{B}(\xi; \hbar) = \mathcal{L}_\omega(\xi) + \varepsilon^2 \mathcal{B}_2(\xi) + \varepsilon^4 \mathcal{B}_4(\xi) + \ldots$$

Indeed:

let $\mathcal{M}$ denote the set of functions $f : \mathbf{T}^l \to \mathbf{C}$ with a definite parity (either even or odd) and $\forall f \in \mathcal{M}$ define

$$Jf = \begin{cases} +1, & \text{if} f \text{ is even,} \\ -1, & \text{if} f \text{ is odd.} \end{cases}$$

Then $J\{f, g\}_M = -(Jf)(Jg)$ and one easily obtains:

$J\mathcal{V}_k = (-1)^k$; $\quad J\mathcal{W}_k = (-1)^{k+1}$, whence

$J\mathcal{V}_{2k+1} = -1$ and $\mathcal{B}_{2k+1} = \mathcal{V}_{2k+1,0} = 0$.

Recovery of the CNF for $\hbar = 0$

Consider the asymptotic expansion of $\mathcal{W}(\xi, x; \hbar)$, $\mathcal{B}(\xi; \hbar)$, $\mathcal{V}(\xi, x; \hbar)$ in powers of $\hbar$ at $\hbar = 0$:

$$\mathcal{W}_k(\xi; x; \hbar) \sim \sum_{j=0}^{\infty} \mathcal{W}_k^{(j)}(\xi, x)\hbar^j$$
$$\mathcal{B}_k(\xi; \hbar) \sim \sum_{j=0}^{\infty} \mathcal{B}_k^{(j)}(\xi)\hbar^j$$
$$\mathcal{V}_k(\xi; x; \hbar) \sim \sum_{j=0}^{\infty} \mathcal{V}_k^{(j)}(\xi, x)\hbar^j.$$

The principal symbols

$w_k := \mathcal{W}_k^{(0)},\ b_k = \mathcal{B}_k^{(0)},\ v_k = \mathcal{V}_k^{(0)}$

coincide with the coefficients of order k of the CNF generated by the Hamiltonian family

$\mathcal{H}_\varepsilon(\xi, x) = \mathcal{L}_\omega(\xi) + \varepsilon \mathcal{V}_\omega(\xi, x)$.

In fact, the recursive homological equations:

$$\{\mathcal{W}_k, \mathcal{L}\}_M + \mathcal{V}_k = \mathcal{B}_k, \qquad \{\mathcal{L}, \mathcal{B}_k\}_M = 0, \quad k = 1, \ldots$$

evaluated at $\hbar = 0$ become:

$$\{w_k, \mathcal{L}\} + v_k = b_k, \qquad \{\mathcal{L}, b_k\} = 0, \qquad \nu_1 \equiv v \equiv \mathcal{V}$$

$$v_k = \sum_{r=2}^{k} \frac{1}{r!} \sum_{\substack{j_1+\ldots+j_r=k \\ j_s \geq 1}} \{w_{j_1}, \{w_{j_2}, \ldots, \{w_{j_r}, \mathcal{L}\} \ldots\}$$

$$+ \sum_{r=1}^{k-1} \frac{1}{r!} \sum_{\substack{j_1+\ldots+j_r=k-1 \\ j_s \geq 1}} \{w_{j_1}, \{w_{j_2}, \ldots, \{w_{j_r}, v\} \ldots\} \quad (9)$$

This is exactly the recurrence defined by **canonical perturbation theory**. Indeed:

Look for an ε-dependent family of smooth canonical maps $\Phi_\varepsilon : \mathbf{R}^l \times \mathbf{T}^l \leftrightarrow \mathbf{R}^l \times \mathbf{T}^l$,

$(\xi, x) \mapsto (\eta, y) = \Phi_\varepsilon(\xi, x)$ such that

$$\mathcal{H}_\varepsilon \circ \Phi_\varepsilon^{-1}(\xi, x) = \mathcal{L}(\xi) + \varepsilon b_1(\xi) + \varepsilon^2 b_2(\xi) + \ldots \quad (10)$$

Look for Φ_ε as the time 1 flow of a smooth Hamiltonian family $w_\varepsilon(\xi, x)$:

generating function. Then

$$\mathcal{H}_\varepsilon \circ \Phi_\varepsilon^{-1}(\xi, x)$$

$$= \mathcal{H}_\varepsilon(\xi, x) + \sum_{s=1}^{\infty} \{w_\varepsilon^{(1)}, \{w_\varepsilon^{(2)}, \dots \{w_\varepsilon^{(s)}, \mathcal{L}\} \dots\} \quad (11)$$

where $w_\varepsilon^{(r)} = w_\varepsilon$, $\forall r = 1, 2, \dots$. If we set

$w_\varepsilon = \varepsilon w_1 + \varepsilon^2 w_2 + \dots$

and equate (10) and (11) we obtain

$$b_k := \{w_k, \mathcal{L}\} + v_k, \quad k \geq 1, \ v_1 \equiv v \equiv \mathcal{V}$$

$$v_k = \sum_{r=2}^{k} \frac{1}{r!} \sum_{\substack{j_1+\dots+j_r=k \\ j_s \geq 1}} \{w_{j_1}, \{w_{j_2}, \dots, \{w_{j_r}, \mathcal{L}\} \dots\}$$

$$+ \sum_{r=1}^{k-1} \frac{1}{r!} \sum_{\substack{j_1+\dots+j_r=k-1 \\ j_s \geq 1}} \{w_{j_1}, \{w_{j_2}, \dots, \{w_{j_r}, v\} \dots\}$$

Condition $\{\mathcal{L}, b_k\} = 0$ follows from the fact that both $\mathcal{L}(\xi)$ and $b_k(\xi)$ do not depend on x.