Exceptional points in the Bose-Einstein condensation of cold dilute gases

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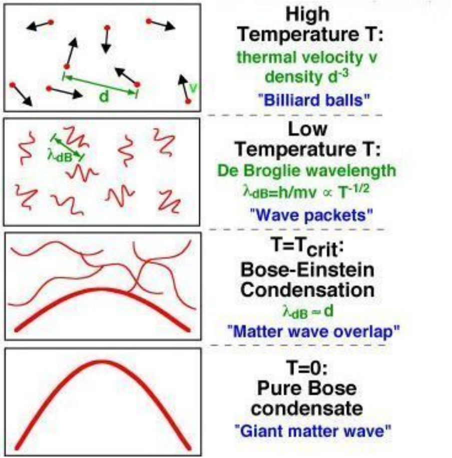
Outline

1. Bose-Einstein condensates
   - Mean-field description and Gross-Pitaevskii equation
   - Condensates with long-range interactions
   - Collapse of a condensate
2. A nonlinear variant of exceptional points
   - Variational ansatz for solving the extended Gross-Pitaevskii equation with an attractive $1/r$ interaction
   - A collapse, a bifurcation and an exceptional point
   - Confirmation with numerically exact solutions to the Gross-Pitaevskii equation
   - Exceptional surface in dipolar Bose-Einstein condensates
3. Summary and outlook

Reminder: Bose-Einstein condensation

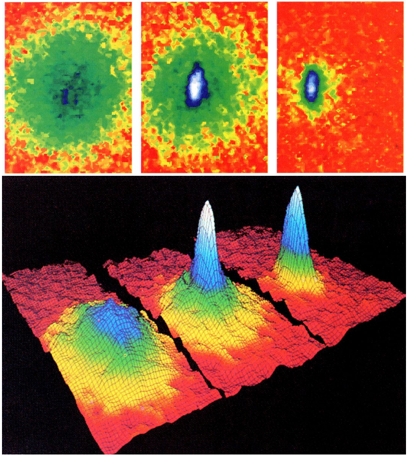
Predicted by Satyendra Nath Bose and Albert Einstein 1924:

When the thermal de Broglie wavelength becomes of the order of the interparticle distance, bosons begin to “condense” into their ground state. All bosons have the same energy and quantum characteristics, similar to the way all photons in a laser share the same quantum state.

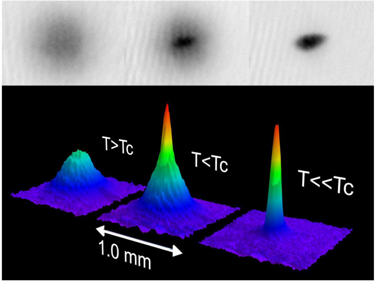


Experimental realisation

BEC in ^{87}Rb by Wieman et al. 1995



BEC in ^{23}Na by Ketterle et al. 1995



Nobel prize 2001

$T = 0\,\mathrm{K}$: Ground state of a BEC

Many-body Hamiltonian of N identical bosons

$$H = \sum_i \frac{\boldsymbol{p}_i^2}{2m} + \sum_i U(\boldsymbol{r}_i) + \sum_{i<j} V(\boldsymbol{r}_i, \boldsymbol{r}_j)$$

- External potential $U(\boldsymbol{r})$
- Two-body interaction potential $V(\boldsymbol{r}, \boldsymbol{r}')$
- Zero-temperature bosonic ground state: $\Psi = \prod_{i=1}^{N} \psi(\boldsymbol{r}_i)$

Hartree equation for single-particle orbital ψ

$$\left\{ \frac{\boldsymbol{p}^2}{2m} + U(\boldsymbol{r}) + (N-1) \int V(\boldsymbol{r}, \boldsymbol{r}') |\psi(\boldsymbol{r}', t)|^2 \, \mathrm{d}^3\boldsymbol{r}' \right\} \psi(\boldsymbol{r}, t) = \mathrm{i}\hbar \frac{\partial \psi(\boldsymbol{r}, t)}{\partial t}$$

Nonlinear Schrödinger equation!

Dilute gases in a harmonic trap

- External trapping potential to confine the condensate:

$$U(\boldsymbol{r}) = \frac{m}{2}\left(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2\right)$$

- Short-range two-body contact interaction (s-wave scattering interaction with scattering length a):

$$V_s(\boldsymbol{r}, \boldsymbol{r}') = \frac{4\pi a \hbar^2}{m}\delta(\boldsymbol{r} - \boldsymbol{r}')$$

- For $N \gg 1$ [i.e. $(N-1) \approx N$]: macroscopic wave function $\Psi(\boldsymbol{r}, t) = \sqrt{N}\psi(\boldsymbol{r}, t)$:

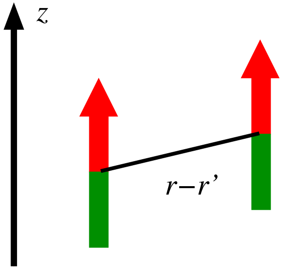
Gross-Pitaevskii equation for Ψ

$$\left\{\frac{\boldsymbol{p}^2}{2m} + \frac{m}{2}\sum_i \omega_i^2 x_i^2 + \frac{4\pi a \hbar^2}{m}|\Psi(\boldsymbol{r}, t)|^2\right\}\Psi(\boldsymbol{r}, t) = \mathrm{i}\hbar\frac{\partial\Psi(\boldsymbol{r}, t)}{\partial t}$$

BEC with long-range dipolar interaction

Experimental realisation of a dipolar BEC:

- Pfau et al., PRL **94**, 160401 (2005)
- chromium (^{52}Cr)
- large magnetic moment: $\mu = 6\mu_{\mathrm{B}}$
- The interaction is
  - long-range
  - anisotropic

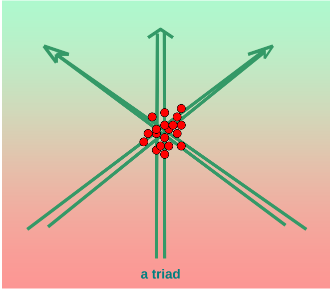


Potential

$$V_{\mathrm{dd}}(\boldsymbol{r},\boldsymbol{r}') = \frac{\mu_0\mu^2}{4\pi}\frac{1-3\cos^2\vartheta'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}$$

BEC with long-range monopolar interaction

Proposal: D.O. O'Dell, et al., PRL **84**, 5697 (2000)

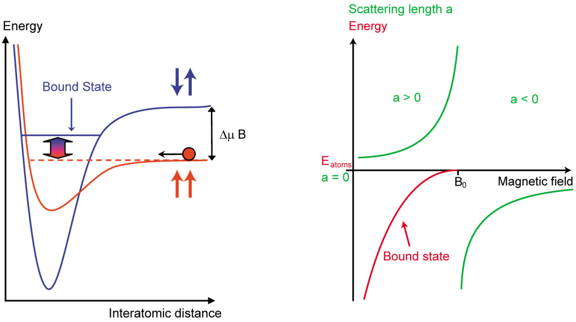


- laser induced dipole-dipole interaction
- 6 "triads" of intense off-resonant laser beams
- $1/r^3$ in the near-zone limit averages out
- gravity-like interaction ("monopolar atoms"):

$$V_u(\boldsymbol{r}, \boldsymbol{r}') = -\frac{u}{|\boldsymbol{r} - \boldsymbol{r}'|}$$

- Novel physical feature: self-trapping of the condensate, without external trap.
- Theoretical advantage: For self-trapping analytical variational calculations are feasible. These will serve as a guide for investigations of more complex situations and condensates.

Tuning of the scattering length via Feshbach resonances



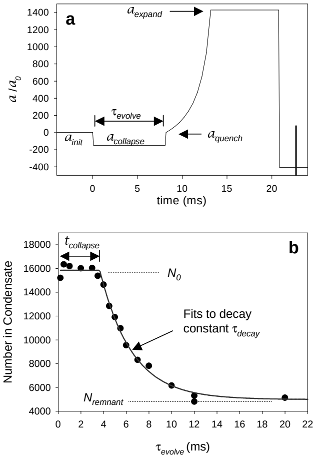
Near B_0

$$a = a_{\mathrm{bg}} \left(1 - \frac{\Delta B}{B - B_0} \right)$$

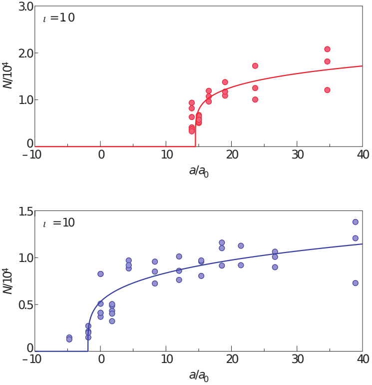
$a > 0$: repulsive, $a < 0$: attractive interaction

Collapse of a condensate at a critical scattering length

Donley et al., Nature **412**, 295 (2001): Collapse of a ^{85}Rb BEC



Koch et al., Nature Physics **4**, 218 (2008): Collapse of a ^{52}Cr BEC



Gross-Pitaevskii equation for a BEC with $1/r$-interaction

- Time-independent extended Gross-Pitaevskii equation with radially symmetric trap in appropriate "atomic units":

$$\varepsilon\psi(\boldsymbol{r}) = \left[-\Delta_{\boldsymbol{r}} + \sum_i \gamma_i^2 x_i^2 + N\left(8\pi a \left|\psi(\boldsymbol{r})\right|^2 - 2\int \mathrm{d}^3\boldsymbol{r}' \frac{\left|\psi(\boldsymbol{r})\right|^2}{|\boldsymbol{r}-\boldsymbol{r}'|}\right)\right]\psi(\boldsymbol{r})$$

- Scaling property:

$$(Nr, N^2 a, \boldsymbol{\gamma}/N^2, \varepsilon/N^2, N^{-3/2}\psi) \quad \longrightarrow \quad (\boldsymbol{r}, a, \boldsymbol{\gamma}, \varepsilon, \psi)$$

- Remaining parameters: $\boldsymbol{\gamma}$, a
- Case of self-trapping: $\boldsymbol{\gamma} = 0$, i.e. there is only one parameter

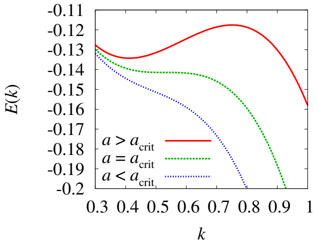
Self-trapping condensate

$$\varepsilon\psi(\boldsymbol{r}) = \left[-\Delta_{\boldsymbol{r}} + \left(8\pi a \left|\psi(\boldsymbol{r})\right|^2 - 2\int \mathrm{d}^3\boldsymbol{r}' \frac{\left|\psi(\boldsymbol{r})\right|^2}{|\boldsymbol{r}-\boldsymbol{r}'|}\right)\right]\psi(\boldsymbol{r})$$

A variational solution for a BEC with $1/r$-interaction

- Variational method: minimisation of the mean field energy:

$$E[\psi] = \int \mathrm{d}^3\boldsymbol{r}\, \psi^*(\boldsymbol{r}) \left(-\Delta_{\boldsymbol{r}} + 4\pi a \left|\psi(\boldsymbol{r})\right|^2 - \int \mathrm{d}^3\boldsymbol{r}' \frac{\left|\psi(\boldsymbol{r})\right|^2}{|\boldsymbol{r}-\boldsymbol{r}'|} \right) \psi(\boldsymbol{r})$$



$$a_{\text{crit}} = -\frac{3\pi}{8}$$

- Trial wave function:

$$\psi(\boldsymbol{r}) = A \exp\left(\frac{-k^2 \boldsymbol{r}^2}{2} \right)$$

- Solution:

$$k_{\pm} = \frac{1}{2}\sqrt{\frac{\pi}{2}}\frac{1}{a}\left(\pm\sqrt{1+\frac{8}{3\pi}a} - 1 \right)$$

Two stationary solutions

- Mean field energy:

$$E_{\pm}(a) = \frac{3\pi}{16}\frac{1}{a^2}\left(\pm\sqrt{1+\frac{8}{3\pi}a}-1\right)^2 + \frac{\pi}{16}\frac{1}{a^2}\left(\pm\sqrt{1+\frac{8}{3\pi}a}-1\right)^3 - \frac{1}{2}\frac{1}{a}\left(\pm\sqrt{1+\frac{8}{3\pi}a}-1\right)$$

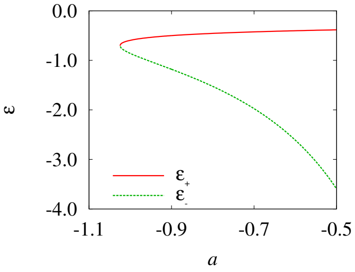
- Chemical potential:

$$\varepsilon_{\pm}(a) = \frac{3\pi}{16}\frac{1}{a^2}\left(\pm\sqrt{1+\frac{8}{3\pi}a}-1\right)^2 + \frac{\pi}{8}\frac{1}{a^2}\left(\pm\sqrt{1+\frac{8}{3\pi}a}-1\right)^3 - \frac{1}{a}\left(\pm\sqrt{1+\frac{8}{3\pi}a}-1\right)$$

Bose-Einstein condensates: tangent bifurcation

Typical scenario

- Stationary solutions to the Gross-Pitaevskii equation exist only in certain areas of the parameter space defining the physics of the condensate.
- At a critical parameter value two solutions are created in a tangent bifurcation.
- At the point of bifurcation the energy eigenvalues (chemical potentials), the mean field energies and the wave functions of both solutions are identical.



Example

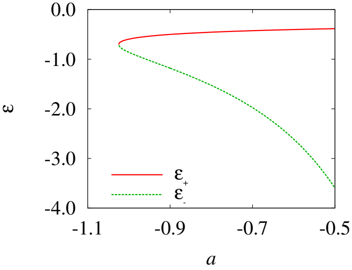
- Condensate with attractive $1/r$ interaction.
- Only one real parameter: scaled scattering length $N^2 a/a_{\mathrm{u}} \to a$.

Is the degeneracy an exceptional point?

Degeneracy

$$a = a_{\text{crit}} \quad \rightarrow \quad \begin{aligned} k_+ &= k_-, \\ E_+ &= E_- \\ \varepsilon_+ &= \varepsilon_-, \end{aligned}$$

- energies are identical
- wave functions ψ_{k_+} and ψ_{k_-} are identical



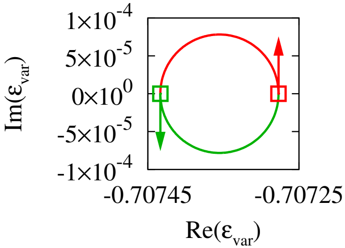
Access to the permutation behaviour:

- A two-dimensional parameter space is required: extension to complex numbers: $a \in \mathbb{C}$
- A clear proof is the permutation of two eigenvalues if a circle around the critical parameter value is traversed:

$$a = a_{\text{crit}} + r\mathrm{e}^{i\varphi}, \qquad \varphi = 0 \ldots 2\pi$$

Circle with a small radius

Numerical result, $r = 10^{-8}$



The surprising(?) result

- A clear permutation of the two solutions for the chemical potential is visible.
- A permutation of the two mean field energy values is not visible.

Is there an explanation for this behaviour?

Circle with a small radius ($r \ll 1$)

Analytic approximation of the energy

Power series expansion

$$E_\pm(\varphi) = -\frac{4}{9\pi} + 0 \cdot \sqrt{r}\mathrm{e}^{\mathrm{i}\varphi/2} + \frac{32}{27\pi^2} \cdot \sqrt{r}^2 \mathrm{e}^{\mathrm{i}\varphi} \pm \left(\frac{4}{9\pi} - \frac{32}{9\pi^2}\right) \cdot \sqrt{r}^3 \mathrm{e}^{(3/2)\mathrm{i}\varphi} + \mathrm{O}\left(\sqrt{r}^4\right)$$

- The first order term with the phase factor $\mathrm{e}^{\mathrm{i}\varphi/2}$ vanishes.
- The lowest non-vanishing order has the phase factor $\mathrm{e}^{\mathrm{i}\varphi}$ which does not lead to a permutation of the energies.
- The third order term is the lowest order responsible for a permutation.

Circle with a small radius ($r \ll 1$)

Analytic approximation of the chemical potential

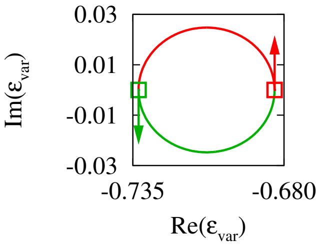
Power series expansion

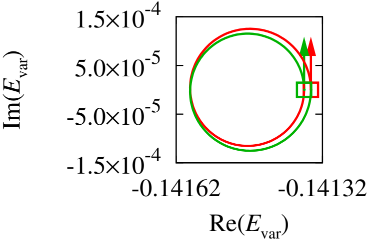
$$\varepsilon_{\pm}(\varphi) = -\frac{20}{9\pi} \pm \frac{8}{3\pi} \cdot \sqrt{r}\mathrm{e}^{\mathrm{i}\varphi/2} - \left(\frac{4}{3\pi} + \frac{128}{27\pi^2}\right) \cdot \sqrt{r}^2 \mathrm{e}^{\mathrm{i}\varphi} \pm \left(\frac{8}{9\pi} - \frac{64}{9\pi^2}\right) \cdot \sqrt{r}^3 \mathrm{e}^{(3/2)\mathrm{i}\varphi} + \mathrm{O}\left(\sqrt{r}^4\right)$$

- The first order term with the phase factor $\mathrm{e}^{\mathrm{i}\varphi/2}$ does not vanish.
- The lowest non-vanishing order leads to a permutation of the chemical potential values.
- The permutation of the eigenvalues appears for a small radius.

Increasing radius

$r = 10^{-3}$

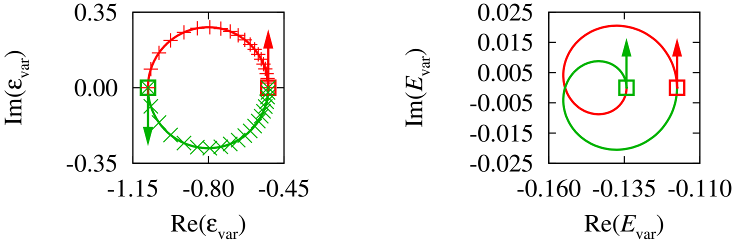




- For increasing radii, higher order terms become more and more important and the permutation of the mean field energy values becomes visible.
- The permutation of the chemical potential solutions does not change.

Deformation of the circle

$r = 10^{-1}$



- For a further increasing radius, a deformation of the circle of the mean field energy appears (as in our linear model system).
- The shape of the chemical potential circle does not change but the spacing between the points is no longer uniform.

Circle with a large radius ($r \gg 1$)

Analytic approximation

- Mean field energy:

$$E_\pm(\varphi) = \pm\left(\frac{\pi}{16} - \frac{1}{2}\right) \cdot \frac{\mathrm{e}^{-\mathrm{i}\varphi/2}}{\sqrt{r}} + \frac{1}{2} \cdot \frac{\mathrm{e}^{-\mathrm{i}\varphi}}{\sqrt{r}^2} \pm \left(\frac{3\pi^2}{64} - \frac{3\pi}{8}\right) \cdot \frac{\mathrm{e}^{-(3/2)\mathrm{i}\varphi}}{\sqrt{r}^3} + \mathrm{O}\left(\frac{1}{\sqrt{r}^4}\right)$$

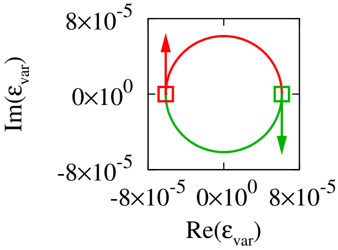
- Chemical potential:

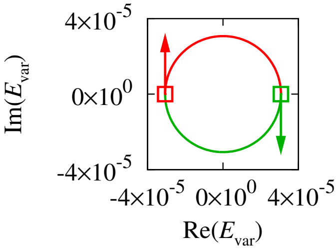
$$\varepsilon_\pm(\varphi) = \pm\left(\frac{\pi}{8} - 1\right) \cdot \frac{\mathrm{e}^{-\mathrm{i}\varphi/2}}{\sqrt{r}} + \left(1 - \frac{3\pi}{16}\right) \cdot \frac{\mathrm{e}^{-\mathrm{i}\varphi}}{\sqrt{r}^2} \pm \left(\frac{3\pi^2}{32} - \frac{3\pi}{8}\right) \cdot \frac{\mathrm{e}^{-(3/2)\mathrm{i}\varphi}}{\sqrt{r}^3} + \mathrm{O}\left(\frac{1}{\sqrt{r}^4}\right)$$

- A clear semicircle structure is expected for a large radius.

Circle with a large radius ($r \gg 1$)

Numerical result, $r = 10^8$





Semicircles are visible for both the mean field energy and the chemical potential.

Open question

So far we investigated only variational solutions. Is the exceptional point a property of the Gross-Pitaevskii equation?

Numerical solution of the Gross-Pitaevskii equation

System of coupled differential equations

$$\Psi''(r) + \frac{2}{r}\Psi'(r) = -U(r)\Psi(r) + 8\pi b|\Psi(r)|^2\Psi(r) \; ,$$
$$U''(r) + \frac{2}{r}U'(r) = -8\pi|\Psi(r)|^2 \; ,$$

- Procedure: Perform a circle in the complex plane around the numerically accurate bifurcation point $a = -1.0251$ and solve the Gross-Pitaevskii equation for each point on the circle.
- The GP equation contains the square modulus of the wave function ψ and is therefore a non-analytic function of ψ.
- Nontrivial task: An analytic continuation of the Gross-Pitaevskii equation is required to obtain the corresponding wave functions.

Analytic continuation of the Gross-Pitaevskii equation

- Any complex wave function can be written as

$$\psi(\boldsymbol{r}) = \mathrm{e}^{\alpha(\boldsymbol{r})+\mathrm{i}\beta(\boldsymbol{r})} \, ,$$

  with real functions $\alpha(\boldsymbol{r})$ and $\beta(\boldsymbol{r})$ to determine the amplitude and phase of the wave function, respectively.
- The complex conjugate and the square modulus of $\psi(\boldsymbol{r})$ read:

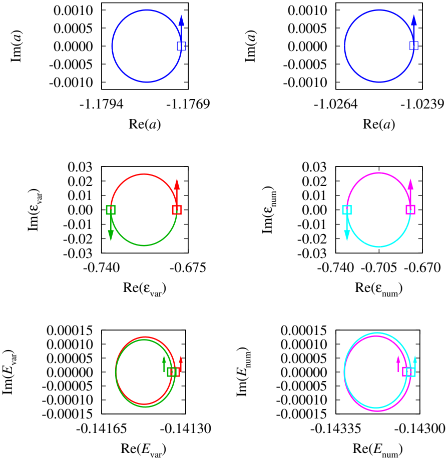
$$\psi^*(\boldsymbol{r}) = \mathrm{e}^{\alpha(\boldsymbol{r})-\mathrm{i}\beta(\boldsymbol{r})} \, , \; |\psi(\boldsymbol{r})|^2 = \mathrm{e}^{2\alpha(\boldsymbol{r})}$$

- The GP system can then be transformed into two coupled nonlinear differential equations for the real functions $\alpha(\boldsymbol{r})$ and $\beta(\boldsymbol{r})$, however, without any complex conjugate or square modulus. These equations can now be continued analytically by allowing for complex valued functions $\alpha(\boldsymbol{r})$ and $\beta(\boldsymbol{r})$.
- Consequence: The square modulus, $|\psi(\boldsymbol{r})|^2 = \mathrm{e}^{2\alpha(\boldsymbol{r})}$ can become complex, leading to a complex absorbing potential in the Gross-Pitaevskii equation.

Comparison of variational and numerically exact results

Encircling the degeneracy

- Qualitatively the same behaviour.
- Quantitative differences which already appear for all stationary solutions.
- The exact numerical calculations confirm the variational result.
- The exceptional point is not an effect of the variational approximation.

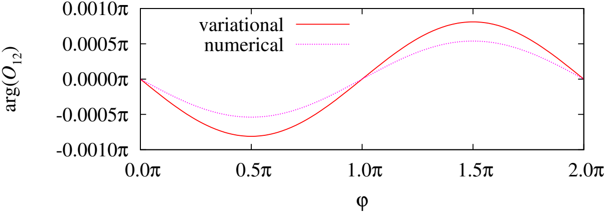


Phase behaviour

- Do the two eigenstates change sign while they permute as a circle in the complex parameter plane is traversed?
- The eigenfunctions of the (nonlinear) stationary Gross-Pitaevskii equation are not orthogonal. A "left-hand" vector cannot clearly be defined.
- An accessible value is the complex overlap integral for the two non-orthogonal normalised states

$$O_{12} = 4\pi \int \psi_1(r)\psi_2(r)\, r^2 \mathrm{d}r$$

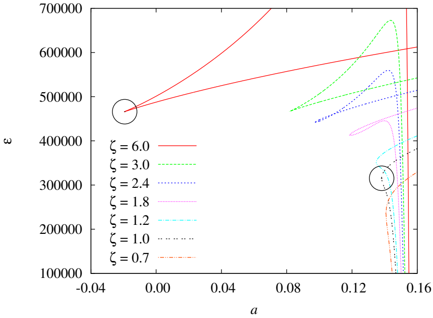
- A phase behaviour without a change in sign is expected. However, a possible true geometric phase is not accessible.



Dipolar Bose-Einstein condensates

Scaled Gross-Pitaevskii equation

$$\left[-\Delta_{\boldsymbol{r}} + \gamma_\varrho^2 \varrho^2 + \gamma_z^2 z^2 + 8\pi a|\psi(\boldsymbol{r})|^2 + \int |\psi(\boldsymbol{r}')|^2 \frac{1-3\cos^2\vartheta'}{|\boldsymbol{r}-\boldsymbol{r}'|^3} \mathrm{d}^3\boldsymbol{r}'\right]\psi(\boldsymbol{r}) = \varepsilon\psi(\boldsymbol{r})$$



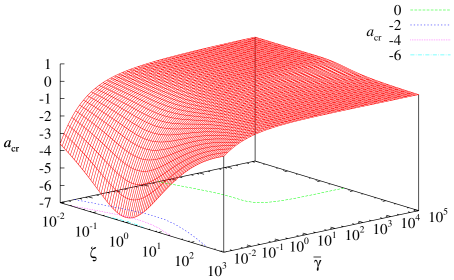
Example

- Mean trap frequency: $\bar{\gamma} = 3.4 \times 10^4$.
- $\zeta = \gamma_z/\gamma_r$.

“Exceptional surface” for dipolar condensates

Three dimensional parameter space

- $\bar{\gamma} = \gamma_r^{2/3} \gamma_z^{1/3}$
- $\zeta = \gamma_z / \gamma_r$
- a
- Complex extension: $a \in \mathbb{C} \to$ four dimensional

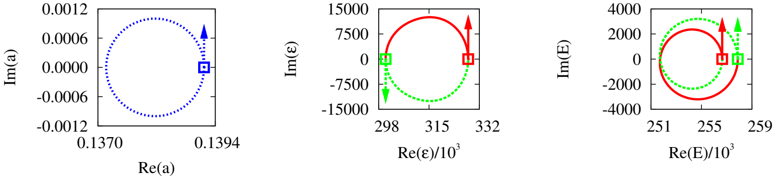


- Generalised exceptional point: co-dimension 2
- Here: two-dimensional object $\to$ surface

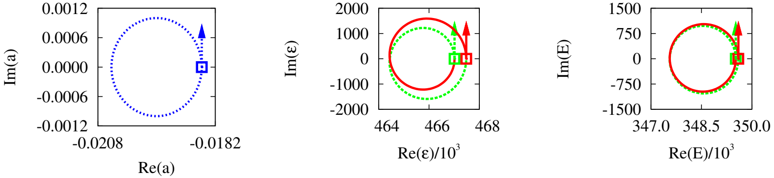
Circle around the degeneracy for dipolar condensates

$$a = a_{\text{crit}} + r e^{i\varphi}, \qquad \varphi = 0 \dots 2\pi$$

- $\lambda = 1$: attractive dipole-dipole interaction



- $\lambda = 6$: repulsive dipole-dipole interaction



Summary

- Bose-Einstein condensates exhibit two stationary states which are born in a tangent bifurcation under variations of the s-wave scattering length.
- The bifurcation points are a nonlinear variant of an exceptional point.
- The identification of the exceptional points is possible with a complex extension of the scattering length leading to complex absorbing potentials.
- They have been found and identified in BEC
    - a harmonic trap
    - with attractive $1/r$ interaction
    - with dipole-dipole interaction
- BEC near the collapse point are experimental realisations of a real physical system close to exceptional points.
- However, a direct experimental proof of the existence of an exceptional point seems not to be possible.

Some questions

- Improving the variational solution for a dipolar BEC: coupled Gaussian wave packets

$$\psi(\boldsymbol{r}) = \sum_k \exp\Big([\boldsymbol{r} - \boldsymbol{q}_k(t)]^{\mathrm{T}} \boldsymbol{A}_k(t) [\boldsymbol{r} - \boldsymbol{q}_k(t)] + \mathrm{i}\boldsymbol{p}_k(t) \cdot [\boldsymbol{r} - \boldsymbol{q}_k(t)] + \gamma_k(t) \Big)$$

- The Gross-Pitaevskii equation is the Hartree equation of a linear many-body Hamiltonian. What is the origin of the exceptional point?

Related articles:

- H. Cartarius, J. Main, G. Wunner, Phys. Rev. A 77, 013618 (2008)
- H. Cartarius, T. Fabčič, J. Main, G. Wunner, Phys. Rev. A 78, 013615 (2008)
- P. Köberle, H. Cartarius, T. Fabčič, J. Main, G. Wunner, New J. Phys. 11, 023017 (2009)