CITY UNIVERSITY LONDON

PT-symmetry breaking in complex nonlinear wave equations and their deformations

Andreas Fring

Quantum Physics with Non-Hermitian Operators
Max Planck Institute, Dresden 15-25 June 2011

based on arXiv:1103.1832 (accepted for publ. in J. Phys. A.)
Andrea Cavaglia (City University), Bijan Bagchi (Calcutta University)

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Integrable models and PT-symmetry

8th UK meeting on Integrable Models, Conformal Field Theory and Related Topics

Edinburgh, 16 & 17 April 2004

Scientific Programme & Timetable | Meeting Arrangements | Registration Form | Participants List

The proposed meeting is to be the eighth in a series of annual one-day meetings on this topic. The main aims of the meeting are:

- The dissemination, explanation and discussion of recent exciting results in this field.
- To promote communication and collaboration within the UK Integrable Models and Conformal Field Theory community, and to bring mathematicians and physicists working in this area together.
- To act as a forum for young researchers to present their work and to become known and integrated into the community.

Speakers:

Carl Bender (Washington)
Richard Blythe (Manchester)
Alexandre Caldeira (Oxford)
Vladimir Dobrev (Newcastle)
Andreas Fring (City)
Yiannis Papadimitriou (Amsterdam)

Integrable models and PT-symmetry

- Calogero-Moser-Sutherland models
  - A. Fring, Mod. Phys. Lett. A21 (2006) 691
  - A. Fring, Acta Polytechnica 47 (2007) 44
  - A. Fring, M. Znojil, J. Phys. A41 (2008) 194010
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  - A. Fring, Pramana J. of Physics 73 (2009) 363
  - A. Fring, M. Smith, J. Phys. A43 (2010) 325201
  - A. Fring, M. Smith, Int. J. of Theor. Phys. 50 (2011) 974
  - talk by M.Smith Thursday 23/06 11:45
- Quantum spin chains
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What is the behaviour of standard quantities in dynamical systems when they are complexified?

Three different scenarios:

- $\mathcal{PT}$-symmetry

$$[\mathcal{PT}, H] = 0 \qquad \text{and} \qquad \mathcal{PT}\Phi = \Phi$$

- spontaneously broken $\mathcal{PT}$-symmetry

$$[\mathcal{PT}, H] = 0 \qquad \text{and} \qquad \mathcal{PT}\Phi \neq \Phi$$

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Quantities of interest:

- energy

$$E = \int_{-a}^{a} \mathcal{H}[u(x)]\, dx = \oint_{\Gamma} \mathcal{H}[u(x)]\, \frac{du}{u_x}$$

- fixed points
- asymptotic behaviour
- k-limit cycles
- bifurcations
- chaos

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The KdV system:

Hamiltonian:

$$\mathcal{H}_{\text{KdV}} = -\frac{\beta}{6}u^3 + \frac{\gamma}{2}u_x^2 \qquad \beta, \gamma \in \mathbb{C}$$

equation of motion:

$$u_t + \beta u u_x + \gamma u_{xxx} = 0$$

Antilinear symmetries:

$$\mathcal{PT}_+ \quad : \quad x \mapsto -x, t \mapsto -t, i \mapsto -i, u \mapsto u \qquad \text{for } \beta, \gamma \in \mathbb{R}$$

$$\mathcal{PT}_- \quad : \quad x \mapsto -x, t \mapsto -t, i \mapsto -i, u \mapsto -u \qquad \text{for } i\beta, \gamma \in \mathbb{R}$$

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- Integrating twice:

$$u_\zeta^2 = \frac{2}{\gamma}\left(\kappa_2 + \kappa_1 u + \frac{c}{2}u^2 - \frac{\beta}{6}u^3\right) =: \lambda P(u)$$

with integration constants $\kappa_1, \kappa_2 \in \mathbb{C}$

- traveling wave: $u(x,t) = u(\zeta)$ with $\zeta = x - ct$
- view this as a 2 dimensional dynamical systems:

$$u_\zeta^R = \pm \mathrm{Re}\left[\sqrt{\lambda}\sqrt{P(u^R + iu^I)}\right]$$
$$u_\zeta^I = \pm \mathrm{Im}\left[\sqrt{\lambda}\sqrt{P(u^R + iu^I)}\right]$$

- the fixed points are the zeros of $P(u)$:

$$u_\zeta^R = 0$$
$$u_\zeta^I = 0$$

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Linearisation at the fixed point u_f:

$$\begin{pmatrix} u^R_\zeta \\ u^I_\zeta \end{pmatrix} = J(u^R, u^I)\Big|_{u=u_f} \begin{pmatrix} u^R_\zeta \\ u^I_\zeta \end{pmatrix}$$

with Jacobian matrix

$$J(u^R, u^I)\Big|_{u=u_f} = \begin{pmatrix} \pm\frac{\partial \,\mathrm{Re}[\sqrt{\lambda}\sqrt{P(u)}]}{\partial u^R} & \pm\frac{\partial \,\mathrm{Re}[\sqrt{\lambda}\sqrt{P(u)}]}{\partial u^I} \\ \pm\frac{\partial \,\mathrm{Im}[\sqrt{\lambda}\sqrt{P(u)}]}{\partial u^R} & \pm\frac{\partial \,\mathrm{Im}[\sqrt{\lambda}\sqrt{P(u)}]}{\partial u^I} \end{pmatrix}\Bigg|_{u=u_f}$$

Linearisation theorem: *Consider a nonlinear system which possesses a simple linearisation at some fixed point. Then in a neighbourhood of the fixed point the phase portraits of the linear system and its linearisation are qualitatively equivalent, if the eigenvalues of the Jacobian matrix have a nonzero real part, i.e. the linearized system is not a centre.*

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The ten similarity classes for J

$j_i \in \mathbb{R}$	$j_1 > j_2 > 0$	unstable node
	$j_2 < j_1 < 0$	stable node
	$j_2 < 0 < j_1$	saddle point
$j_1 = j_2$, diagonal J	$j_i > 0$	unstable star node
	$j_i < 0$	stable star node
$j_1 = j_2$, nondiagonal J	$j_i > 0$	unstable improper node
	$j_i < 0$	stable improper node
$j_i \in \mathbb{C}$	$\mathrm{Re}\, j_i > 0$	unstable focus
	$\mathrm{Re}\, j_i = 0$	centre
	$\mathrm{Re}\, j_i < 0$	stable focus

Further integration:

$$\pm\sqrt{\lambda}\,(\zeta-\zeta_0)=\int du\frac{1}{\sqrt{P(u)}}$$

assume: $P(u)=(u-A)^3$, which is possible for

$$\lambda=-\frac{\beta}{3\gamma},\quad \kappa_1=-\frac{c^2}{2\beta},\quad \kappa_2=\frac{c^3}{6\beta^2}\quad \text{and}\quad A=\frac{c}{\beta}$$

then:

$$u(\zeta)=\frac{c}{\beta}-\frac{12\gamma}{\beta\,(\zeta-\zeta_0)^2}$$

energy:

$$E_a=-\frac{ac^2}{3\beta^2}\left(c+\frac{36\gamma}{a^2-\zeta_0^2}\right)+\frac{72\gamma^2}{15\beta^2}\left[\frac{10c\left(a^3+3a\zeta_0^2\right)}{\left(a^2-\zeta_0^2\right)^3}-\frac{48\gamma\left(a^5+10a^3\zeta_0^2+5a\zeta_0^4\right)}{\left(a^2-\zeta_0^2\right)^5}\right]$$

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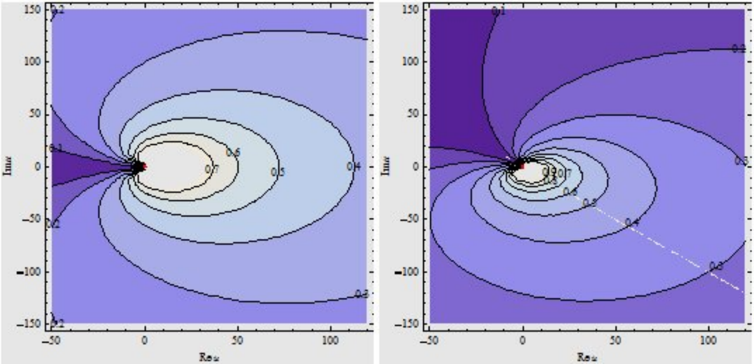
$\mathcal{PT}$-symmetric, spontaneously broken and broken solutions (rational)



(a) $\mathcal{PT}$-symmetric: $c = 1$, $\beta = 2$, $\gamma = 3$, $A = 1/2$
(b) broken $\mathcal{PT}$-symmetry: $c = 1$, $\beta = 2 + i2$, $\gamma = 3$, $A = \frac{1-i}{4}$

The energy is real for (a) and complex for (b).

assume: $P(u) = (u - A)^2(u - B)$, which is possible for

$$\lambda = -\frac{\beta}{3\gamma}, \quad \kappa_1 = \frac{A}{2}(\beta A - 2c), \quad \kappa_2 = \frac{A^2}{6}(3c - 2\beta A), \quad B = \frac{3c}{\beta} - 2A$$

then (with one free parameter):

$$u(\zeta) = B + (A - B)\tanh^2\left[\frac{1}{2}\sqrt{A - B}\sqrt{\lambda}(\zeta - \zeta_0)\right]$$

linearisation: $(A - B = r_{AB}e^{i\theta_{AB}}, \lambda = r_\lambda e^{i\theta_\lambda})$

$$J(A) = \begin{pmatrix} \pm\sqrt{r_{AB}r_\lambda}\cos\left[\frac{1}{2}(\theta_{AB} + \theta_\lambda)\right] & \mp\sqrt{r_{AB}r_\lambda}\sin\left[\frac{1}{2}(\theta_{AB} + \theta_\lambda)\right] \\ \pm\sqrt{r_{AB}r_\lambda}\sin\left[\frac{1}{2}(\theta_{AB} + \theta_\lambda)\right] & \pm\sqrt{r_{AB}r_\lambda}\cos\left[\frac{1}{2}(\theta_{AB} + \theta_\lambda)\right] \end{pmatrix}$$

with eigenvalues ($\in i\mathbb{R}$ for $A < B, \lambda > 0$ or $A > B, \lambda < 0$)

$$j_1 = \pm\sqrt{r_{AB}r_\lambda}\exp\left[\frac{i}{2}(\theta_{AB} + \theta_\lambda)\right]$$

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$$j_2 = \pm\sqrt{r_{AB}r_\lambda}\exp\left[-\frac{i}{2}(\theta_{AB}+\theta_\lambda)\right]$$

Energy for the periodic motion for one period:

$$E_T = \oint_\Gamma \mathcal{H}\left[u(\zeta)\right] \frac{du}{u_\zeta} = \oint_\Gamma \frac{\mathcal{H}\left[u\right]}{\sqrt{\lambda}\sqrt{u-B}(u-A)} du = -\pi\sqrt{\frac{\beta\gamma}{3}} \frac{A^3}{\sqrt{A-B}}$$

In general:

- $E_T \in \mathbb{R}$ for $\mathcal{PT}$-symmetric solution
- $E_T \in \mathbb{C}$ for spontaneously broken $\mathcal{PT}$-symmetric solution
- $E_T \in \mathbb{C}$ for broken $\mathcal{PT}$-symmetric solution

But:

$$E_T \in \mathbb{R} \quad \text{for } A = \frac{\sin\theta_\gamma}{|\beta| \sin\left(\theta_\gamma - 2\theta_\beta/3\right)} \exp\left(-i\frac{\theta_\beta}{3}\right).$$

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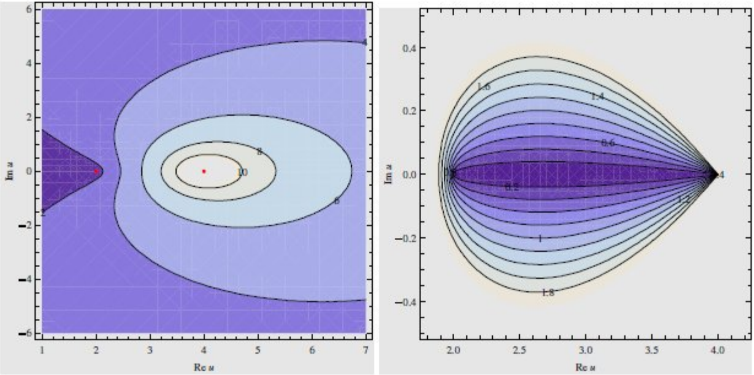
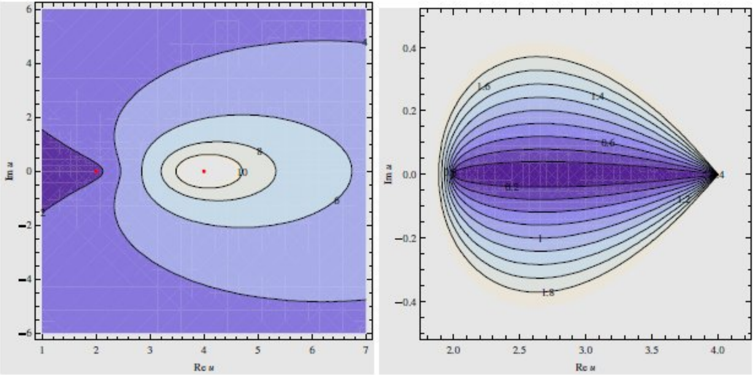
$\mathcal{PT}$-symmetric solution:



(a) periodic: $c = 1$, $\beta = 3/10$, $\gamma = 3$, $A = 4$, $B = 2$, $T = 2\sqrt{15}\pi$
(b) asympt. constant: $c = 1$, $\beta = 3/10$, $\gamma = -3$, $A = 4$, $B = 2$

$E_T \in \mathbb{R}$

$\mathcal{PT}$-symmetric solution:



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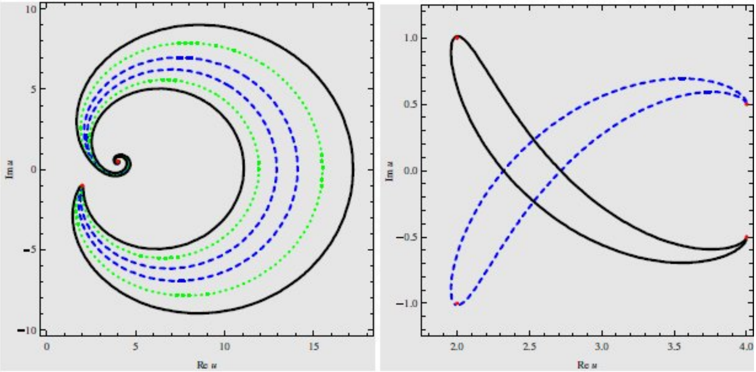
$E_T \in \mathbb{R}$

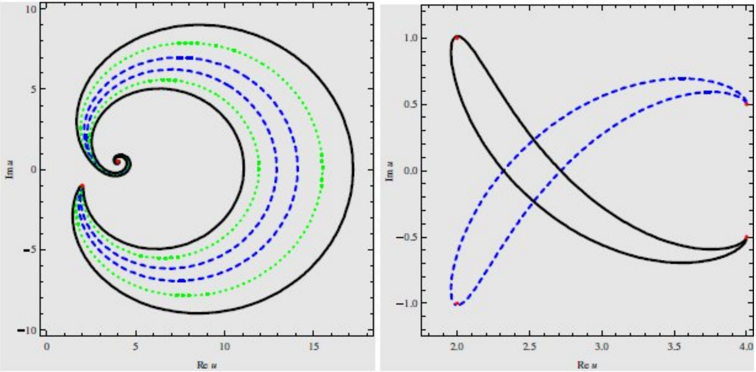
spontaneously broken $\mathcal{PT}$-symmetric solution:



(a) periodic: $c = 1$, $\beta = \frac{3}{10}$, $\gamma = 3$, $A = 4 + \frac{i}{2}$ and $B = 2 - i$ for $\operatorname{Im}\zeta_0 = 0.5$ black, $\operatorname{Im}\zeta_0 = 0.3$ green $\operatorname{Im}\zeta_0 = 0.1$ blue
(b) asympt. constant: $c = 1$, $\beta = \frac{3}{10}$, $\gamma = -3$ for $A = 4 - \frac{i}{2}$, $B = 2 + i$ $\operatorname{Im}\zeta_0 = -0.5$ black; $A = A^*$, $B = B^*$, $\operatorname{Im}\zeta_0 = 0.5$ blue

$E_T \in \mathbb{C}$

spontaneously broken $\mathcal{PT}$-symmetric solution:



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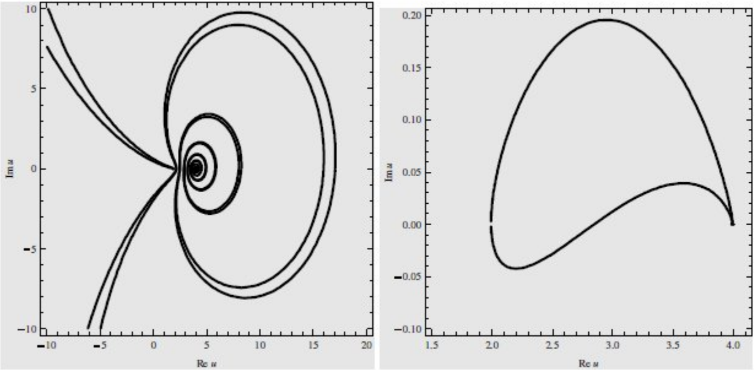
$E_T \in \mathbb{C}$

broken $\mathcal{PT}$-symmetric solution:

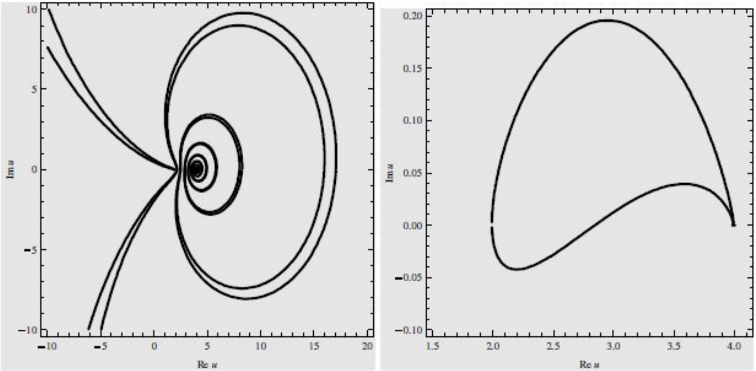


(a) periodic: $A = 4$, $B = 2$, $c = 1$, $\beta = \frac{3}{10}$, $\gamma = 3 + \frac{i}{2}$, $\operatorname{Im} \zeta_0 = 6$

(b) asympt. constant: $A = 4$, $B = 2$, $c = 1$, $\beta = \frac{3}{10}$, $\gamma = -3 + \frac{i}{2}$, $\operatorname{Im} \zeta_0 = 1/2$

$E_T \in \mathbb{C}$

broken $\mathcal{PT}$-symmetric solution:



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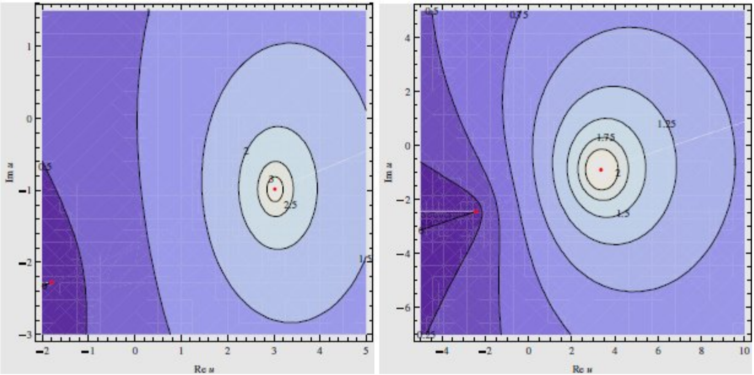
$E_T \in \mathbb{C}$

broken $\mathcal{PT}$-symmetric solution:



(a) periodic solution with complex energy $E_T = -10.52 + i1.67$
(b) periodic solution with real energy $E_T = -4\pi$

assume: $P(u) = (u - A)(u - B)(u - C)$, which is possible for

$$\lambda = -\frac{\beta}{3\gamma}, \quad \kappa_1 = \frac{1}{6}\left[\beta(A^2 + AC + C^2) - 3c(A - C)\right]$$
$$\kappa_2 = \frac{AC}{6}[3c - \beta(A + C)] \quad \text{and} \quad B = \frac{3c}{\beta} - (A + C)$$

then (with two free parameter):

$$u(\zeta) = A + (B - A)\,\mathrm{ns}^2\left[\frac{1}{2}\sqrt{B - A}\sqrt{\lambda}(\zeta - \zeta_0)\left|\frac{A - C}{A - B}\right.\right]$$

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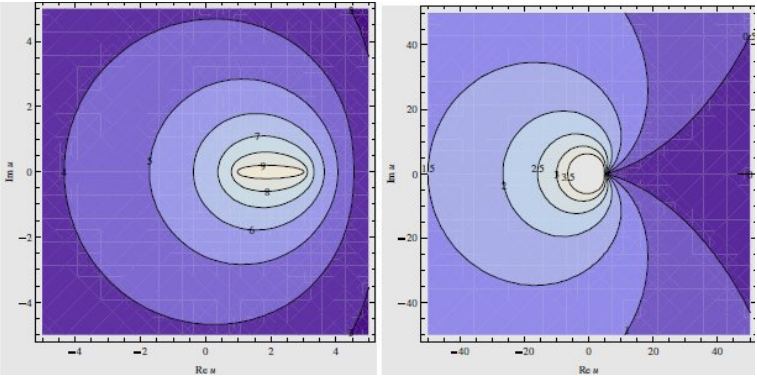
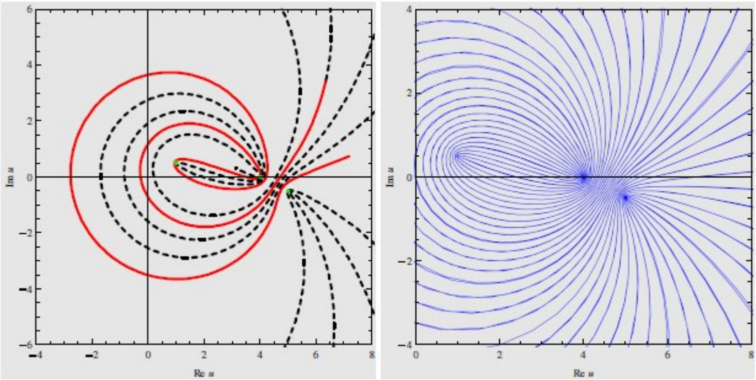
$\mathcal{PT}$-symmetric solution:



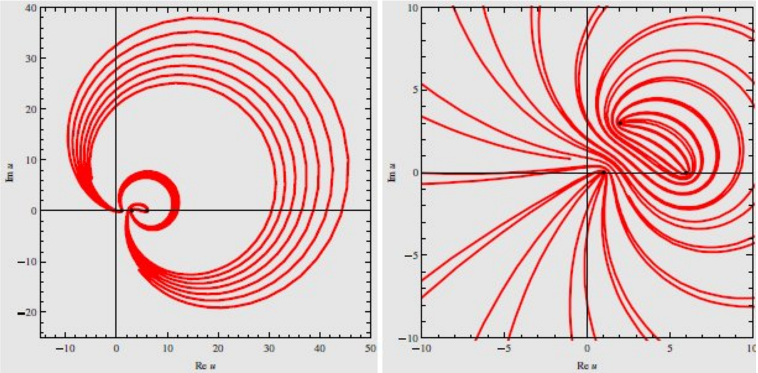
$$A = 1,\ B = 3,\ C = 6,\ c = 1,\ \beta = 3/10,\ \gamma = -3$$

spontaneously broken $\mathcal{PT}$-symmetric solution:



(a) $-64 \leq \zeta \leq 18$ solid (red) and $18 < \zeta \leq 200$ dashed (black)
(b) $-200 < \zeta < 1400$

broken $\mathcal{PT}$-symmetric solution:



(a) $A = 1$, $B = 3$, $C = 6$, $c = 1$, $\beta = 3/10$ and $\gamma = 3 + 2i$ for $-200 \leq \zeta \leq 200$;
(b) $A = 1$, $B = 2 + 3i$, $C = 6$, $c = 1$, $\beta = 3/10 - i/10$ and $\gamma = 3$ for $-200 \leq \zeta \leq 200$

Reduction to quantum mechanical Hamiltonians:

For instance:

$$u \to x, \quad \zeta \to t, \quad \kappa_1 = 0, \quad \kappa_2 = \gamma E, \quad \beta = 6cg, \quad \gamma = -c$$

converts

$$u_\zeta^2 = \frac{2}{\gamma}\left(\kappa_2 + \kappa_1 u + \frac{c}{2}u^2 - \frac{\beta}{6}u^3\right)$$

into Newton's equations for

$$H = E = \frac{1}{2}p^2 + \frac{1}{2}x^2 - gx^3$$

treated in

[C. Bender, D. Brody, D. Hook, Phys. A41 (2008) 352003]

Soliton solutions:

Hirota's bilinear method ($u(x,t) = \frac{12\gamma}{\beta}(\ln\tau)_{xx}$)

$$\frac{6\gamma}{\beta}\left(\gamma D_x^4 + D_x D_t\right)\tau\cdot\tau = 0$$

one soliton solution:

$$u(x,t) = \frac{3\gamma p_1^2}{\beta\cosh^2\left[\frac{1}{2}(p_1 x - \gamma p_1^3 t + \phi_1)\right]}$$

two soliton solution:

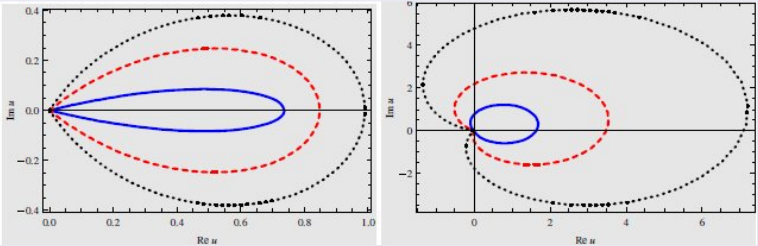
$$u(x,t) = \frac{24\gamma\sum_{k=0}^{6} c_k(-1)^k p_2^k p_1^{6-k}}{\beta(p_1+p_2)^4\left[2\cosh\left(\frac{1}{2}(\eta_1-\eta_2)\right) + e^{-\frac{\eta_1}{2}-\frac{\eta_2}{2}}\left(\frac{e^{\eta_1+\eta_2}(p_1-p_2)^4}{(p_1+p_2)^4}+1\right)\right]^2}$$

where we abbreviated $\eta_i = p_i x - \gamma p_i^3 t + \phi_i$ for $i = 1, 2$ with

$$c_0 = 1+\cosh\eta_2, \quad c_1 = 4\sinh\eta_2, \quad c_2 = \cosh\eta_1 + 6\cosh\eta_2 - 1, \quad c_3 = 4(\sinh\eta_1 + \sinh\eta_2)$$

and $c_i(\eta_1,\eta_2) = c_{6-i}(\eta_2,\eta_1)$

$\mathcal{PT}$-symmetric complex one-soliton solution



(a) $\mathcal{PT}$-symmetric solution with $\beta = 6$, $\gamma = 1$, $p_1 = 1.2$ for $\phi = i0.3$ blue, $\phi = i0.8$ red, $\phi = i1.1$ black, $t = -2$
(b) Broken $\mathcal{PT}$-symmetric solution $\beta = 6$, $\gamma = 1 + i0.4$, $p_1 = 1.2$ for $\phi = i0.3$ blue, $\phi = i0.8$ red, $\phi = i1.1$ black $t = -2$

$\mathcal{PT}$-symmetric complex one-soliton solution

$\beta = 6$, $\gamma = 1$, $p_1 = 1.2$, $\phi = i0.3$,

Complex one-soliton solution with broken $\mathcal{PT}$-symmetry

$\beta = 6$, $\gamma = 1 + i0.4$, $p_1 = 1.2$, $\phi = i0.3$, We obtain a breather regaining its shape when:

$$u\left(x+\Delta_x,t\right)=u\left(x,t+\Delta_t\right)$$

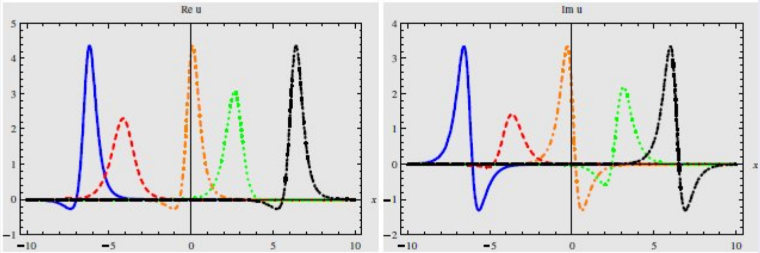
with

$$\begin{aligned}\Delta_t &= \frac{2\pi p_r}{\left(p_i^4-p_r^4\right)\gamma_i-2p_ip_r\left(p_i^2+p_r^2\right)\gamma_r}\\ \Delta_x &= 2\pi\frac{p_i\left(3p_r^2-p_i^2\right)\gamma_i+2\pi p_r\left(3p_i^2-p_r^2\right)\gamma_r}{\left(p_i^4-p_r^4\right)\gamma_i-2p_ip_r\left(p_i^2+p_r^2\right)\gamma_r}\end{aligned}$$

speed of the soliton:

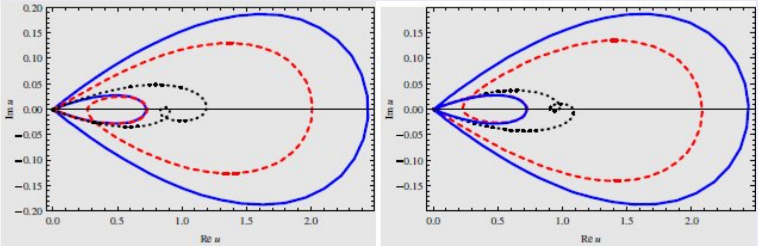
$$v=-\frac{\Delta_x}{\Delta_t}=\left(3p_i^2-p_r^2\right)\gamma_r-\frac{p_i\left(p_i^2-3p_r^2\right)\gamma_i}{p_r}$$

Complex one-soliton solution with broken $\mathcal{PT}$-symmetry



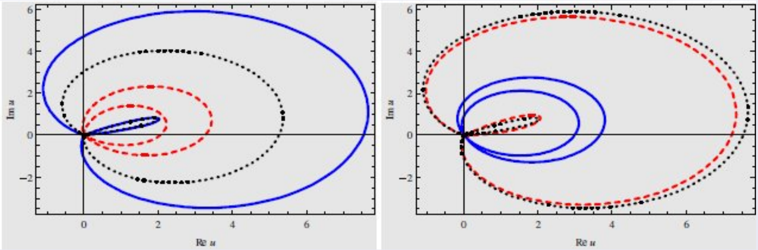
$\beta = 6$, $\gamma = 1 + i/2$, $p_1 = 2$, $\phi = i0.8$ and $\Delta_t = -\pi/2$ for different times $t = -\pi/2$ solid (blue), $t = -1$ dashed (red), $t = 0$ dasheddot (orange), $t = 0.7$ dotted (green), and $t = \pi/2$ dasheddotdot (black) (a) real part; (b) imaginary part

$\mathcal{PT}$-symmetric two soliton solution



$\beta = 6$, $\gamma = 1$, $p_1 = 1.2$, $p_2 = 2.2$, $\phi_1 = i0.1$ and $\phi_2 = i0.2$. (a) $t = -2$ solid (blue), $t = -0.2$ dashed (red), $t = 0.2$ dotted (black); (b) $t = 0.3$ dotted (black), $t = 0.8$ dashed (red), $t = 2.0$ solid (blue)

Two soliton solution with broken $\mathcal{PT}$-symmetry



$\beta = 6$, $\gamma = 1 + i\pi/8$, $p_1 = 2(2/3)^{1/3}$, $p_2 = 2$, $\phi_1 = i0.1$ and $\phi_2 = i0.2$. (a) $t = -4$ solid (blue), $t = -3.5$ dashed (red), $t = -2$. dotted (black); (b) $t = 0.7$ solid (blue), $t = 2$ dashed (red), $t = 8$ dotted (black)

$\Delta_t^1 = -3$, $\Delta_t^2 = -2$,

$\mathcal{PT}$-symmetric complex two-soliton solution

Real part for: $\beta = 6$, $\gamma = 1$, $p_1 = 1.2$, $p_2 = 2.2$, $\phi_1 = i0.1$, $\phi_2 = i0.2$

Complex two-soliton solution with broken $\mathcal{PT}$-symmetry

Real part for: $\beta = 6$, $\gamma = 1 + i\pi/8$, $p_1 = 2(2/3)^{1/3}$, $p_2 = 2$, $\phi_1 = i0.1$ and $\phi_2 = i0.2$

Energy for the one-soliton:

$$E_{1s} = -\frac{36\gamma^3 p_1^5}{5\beta^2}$$

Energy for the two-soliton:

- $\mathcal{PT}$-symmetric case:

$$E_{2s} \approx -10.8049 = E_{1s}(p_1) + E_{1s}(p_2)$$

- Broken $\mathcal{PT}$-symmetric case:

$$E_{2s} \approx -7.8876 - i9.4327 = E_{1s}(p_1) + E_{1s}(p_2)$$

General deformation prescription:

$\mathcal{PT}$-anti-symmetric quantities:

$$\mathcal{PT}: \phi(x,t) \mapsto -\phi(x,t) \quad \Rightarrow \quad \delta_\varepsilon : \phi(x,t) \mapsto -i[i\phi(x,t)]^\varepsilon$$

Two possibilities for the KdV Hamiltonian

$$\delta_\varepsilon^+ : u_x \mapsto u_{x,\varepsilon} := -i(iu_x)^\varepsilon \qquad \text{or} \qquad \delta_\varepsilon^- : u \mapsto u_\varepsilon := -i(iu)^\varepsilon,$$

such that

$$\mathcal{H}_\varepsilon^+ = -\frac{\beta}{6}u^3 - \frac{\gamma}{1+\varepsilon}(iu_x)^{\varepsilon+1} \qquad \mathcal{H}_\varepsilon^- = \frac{\beta}{(1+\varepsilon)(2+\varepsilon)}(iu)^{\varepsilon+2} + \frac{\gamma}{2}u_x^2$$

with equations of motion

$$u_t + \beta u u_x + \gamma u_{xxx,\varepsilon} = 0 \qquad u_t + i\beta u_\varepsilon u_x + \gamma u_{xxx} = 0$$

The $\mathcal{H}_\varepsilon^+$-models

Integrating twice yields now:

$$u_\zeta^{(n)} = \exp\left[\frac{i\pi}{2(\varepsilon+1)}(1-\varepsilon+4n)\right]\left[\lambda_\varepsilon P(u)\right]^{\frac{1}{1+\varepsilon}}$$

Again we can construct systematically solutions by assuming:

$P(u) = (u-A)^3$,
$P(u) = (u-A)^2(u-B)$,
$P(u) = (u-A)(u-B)(u-C)$

but now we have branch cuts.

For instance:

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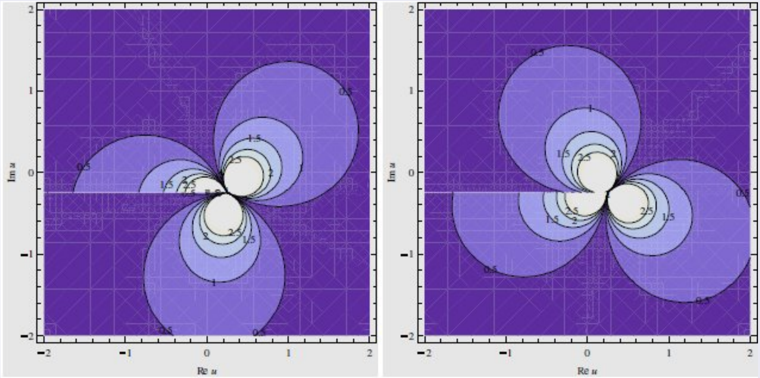
but now we have branch cuts.

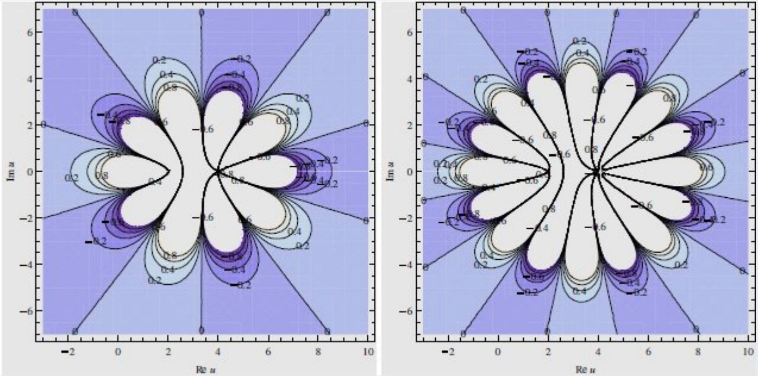
For instance:

Broken $\mathcal{PT}$-symmetric rational solutions for $\mathcal{H}_{1/3}^+$



Different Riemann sheets for $A = (1 - i)/4$, $c = 1$, $\beta = 2 + 2i$ and $\gamma = 3$
(a) $u^{(1)}$
(b) $u^{(2)}$

$\mathcal{PT}$-symmetric trigonometric/hyperbolic solutions

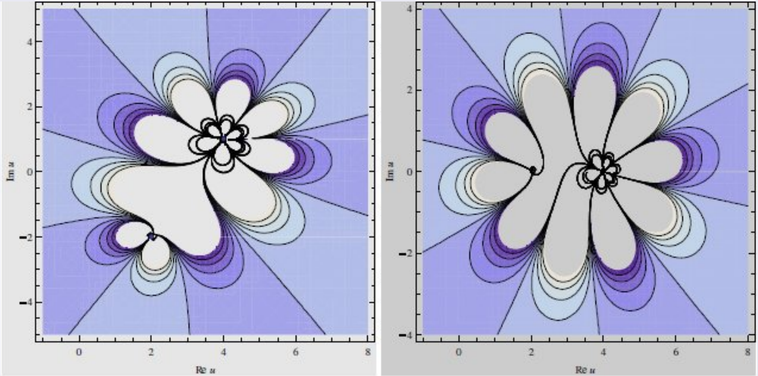


$A = 4, B = 2, c = 1, \beta = 2$ and $\gamma = 3$

(a) $\mathcal{H}^+_{-1/2}$

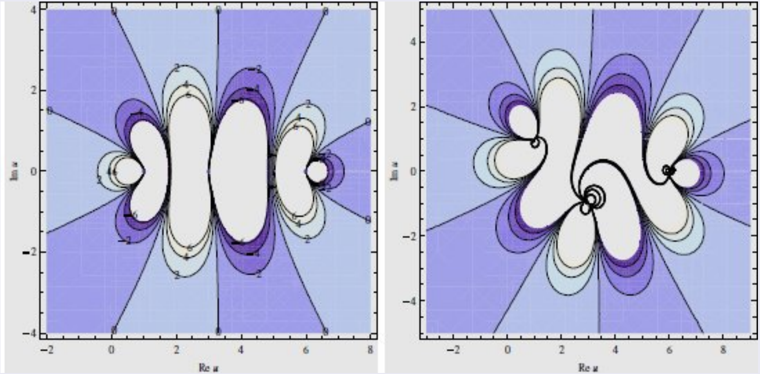
(b) $\mathcal{H}^+_{-2/3}$

Broken $\mathcal{PT}$-symmetric trigonometric solutions for $\mathcal{H}^+_{-1/2}$



(a) Spontaneously broken $\mathcal{PT}$-symmetry with $A = 4 + i$, $B = 2 - 2i$, $c = 1$, $\beta = 3/10$ and $\gamma = 3$
(b) broken $\mathcal{PT}$-symmetry with $A = 4$, $B = 2$, $c = 1$, $\beta = 3/10$ and $\gamma = 3 + i$

Elliptic solutions for $\mathcal{H}_{-1/2}^{+}$:



(a) $\mathcal{PT}$-symmetric with $A = 1$, $B = 3$, $C = 6$, $\beta = 3/10$, $\gamma = -3$ and $c = 1$
(b) spontaneously broken $\mathcal{PT}$-symmetry with $A = 1 + i$, $B = 3 - i$, $C = 6$, $\beta = 3/10$, $\gamma = -3$ and $c = 1$

The $\mathcal{H}_\varepsilon^-$-models

Integrating twice gives now:

$$u_\zeta^2 = \frac{2}{\gamma}\left(\kappa_2 + \kappa_1 u + \frac{c}{2}u^2 - \beta\frac{i^\varepsilon}{(1+\varepsilon)(2+\varepsilon)}u^{2+\varepsilon}\right) =: \lambda Q(u)$$

where

$$\lambda = -\frac{2\beta i^\varepsilon}{\gamma(1+\varepsilon)(2+\varepsilon)}$$

For $\kappa_1 = \kappa_2 = 0$

$$u(\zeta) = \left(\frac{c(\varepsilon+1)(\varepsilon+2)}{i^\varepsilon\beta\left[\cosh\left(\frac{\sqrt{c}\varepsilon(\zeta-\zeta_0)}{\sqrt{\gamma}}\right)+1\right]}\right)^{1/\varepsilon}$$

- $\mathcal{H}_{2}^{-}$:
  $\equiv$ complex version of the modified KdV-equation
- $\mathcal{H}_{4}^{-}$:
  assume $Q(u) = u^2(u^2 - B^2)(u^2 - C^2)$, possible for

$$\kappa_1 = \kappa_2 = 0, \qquad B = iC \qquad \text{and} \qquad C^4 = \frac{15c}{\beta}$$

eigenvalues of Jacobian:

$$\begin{aligned} j_1 &= \pm i\sqrt{r_\lambda} r_B^2 \exp\left[\frac{i}{2}(4\theta_B + \theta_\lambda)\right] \\ j_2 &= \mp i\sqrt{r_\lambda} r_B^2 \exp\left[-\frac{i}{2}(4\theta_B + \theta_\lambda)\right] \end{aligned}$$

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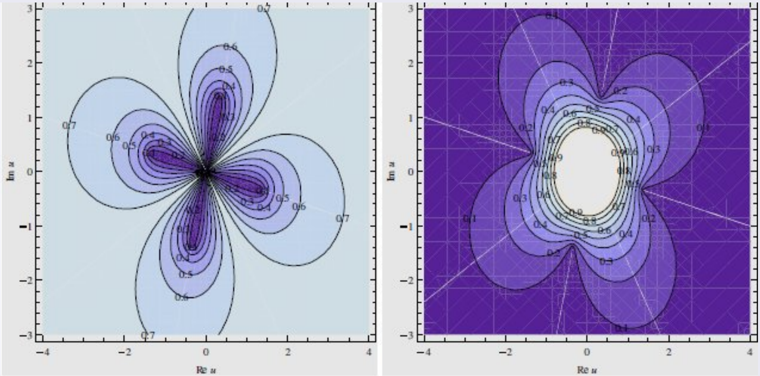
Broken $\mathcal{PT}$-symmetric solution for $\mathcal{H}_4^-$:



(a) star node at the origin for $c = 1$, $\beta = 2 + i3$, $\gamma = 1$ and $B = (15/2 + i3)^{1/4}$
(b) centre at the origin for $c = 1$, $\beta = 2 + i3$, $\gamma = -1$ and $B = (30/13 - i45/13)^{1/4}$

Reduction to quantum mechanical Hamiltonians:
Again we can relate to simple quantum mechanical models:
The identification

$$u \to x, \quad \zeta \to t, \quad \kappa_1 = 0, \quad \kappa_2 = \gamma E, \quad \text{and} \quad \beta = \gamma g(1+\varepsilon)(2+\varepsilon)$$

relates $\mathcal{H}_\varepsilon^-$ to

$$H = E = \frac{1}{2}p^2 - \frac{c}{2\gamma}x^2 + gx^2(ix)^\varepsilon$$

For $c = 0$ these are the "classical models" studied in
[C. Bender, S. Boettcher, Phys. Rev. Lett. 80 (1998) 5243]

Reduction of the $\mathcal{H}_2^-$-model

$$\mathcal{H}_2^-[u] = \frac{\beta}{12}u^4 + \frac{\gamma}{2}u_x^2$$

Twice integrated equation of motion:

$$u_\zeta^2 = \frac{2}{\gamma}\left(\kappa_2 + \kappa_1 u + \frac{c}{2}u^2 + \beta\frac{1}{12}u^4\right) =: \lambda Q(u)$$

Reduction $u \to x, \zeta \to t$

$$\kappa_1 = -\gamma\tau, \quad \kappa_2 = \gamma E_x, \quad \beta = -3\gamma g \quad \text{and} \quad c = -\gamma\omega^2$$

Quartic harmonic oscillator of the form

$$H = E_x = \frac{1}{2}p^2 + \tau x + \frac{\omega^2}{2}x^2 + \frac{g}{4}x^4$$

Boundary cond.: $\kappa_1 = \tau = 0, \lim_{\zeta\to\infty} u(\zeta) = 0, \lim_{\zeta\to\infty} u_x(\zeta) = \sqrt{2E_x}$

[A.G. Anderson, C. Bender, U. Morone, arXiv:1102.4822]

Note: $E_x \neq E_u(a)$

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Note: $E_x \neq E_u(a)$

Assuming: $Q(u) = (u - A)^2 (u - B)(u - C)$

$$u(\zeta) = A + \frac{3(\vartheta - 2c)}{\vartheta e^{\sqrt{\frac{\vartheta - 2c}{\gamma}}(\zeta - \zeta_0)} - A\beta - e^{-\sqrt{\frac{\vartheta - 2c}{\gamma}}(\zeta - \zeta_0)} \beta/8}$$

$\vartheta := 3c + \beta A^2$

Reduced solution:

$$\vartheta = 0 \qquad E_x = -\frac{\omega^4}{4g} \quad \text{and} \qquad A = i\frac{\omega}{\sqrt{g}}$$

$$x(t) = \frac{\omega}{\sqrt{-g}} \tanh\left[\frac{\omega(t + t_0)}{\sqrt{2}}\right]$$

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Linearisation about the fixed point A:
Eigenvalues of the Jacobian matrix

$$j_1 = \pm r_A \sqrt{r_\lambda} \exp\left[\frac{i}{2}(2\theta_A + \theta_\lambda)\right] \quad j_2 = \pm r_A \sqrt{r_\lambda} \exp\left[-\frac{i}{2}(2\theta_A + \theta_\lambda)\right]$$

Recall: $E_x = -\frac{\omega^4}{4g}$, $\lambda = \frac{\beta}{6\gamma}$
Condition for A to be a centre: $2\theta_A + \theta_\lambda = \pi$
Condition for E_x to be real: $4\theta_\omega - \theta_g = 0, \pi$

All possible scenarios exist:

periodic orbits with real energies	for $\omega \in i\mathbb{R}, g \in \mathbb{R}$
periodic orbits with nonreal energies	for $\omega \in i\mathbb{R}, g \notin \mathbb{R}$
nonperiodic orbits with real energies	for $\omega \notin i\mathbb{R}, \omega^4/g \in \mathbb{R}$
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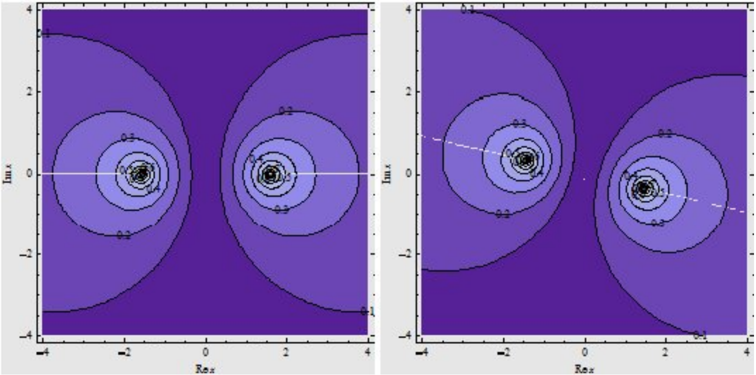
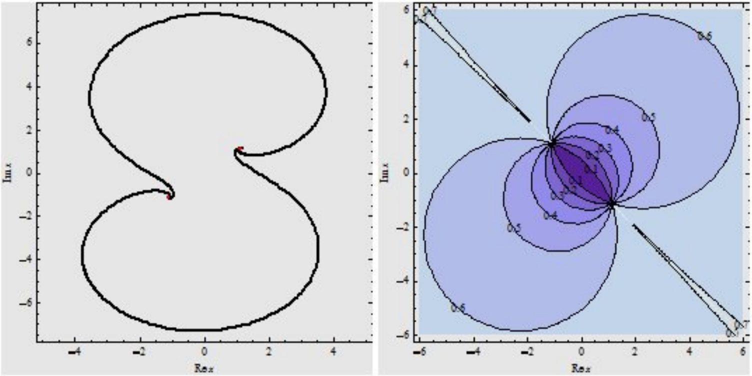
The quartic harmonic oscillator from complex modified KDV



(a) Periodic orbits $E = -25/4$ for $g = 4$, $\omega = i\sqrt{10}$
(b) Periodic orbits $E = -5 + i5/2$ for $g = 4 + 2i$, $\omega = i\sqrt{10}$



(a) Nonperiodic orbits $E = -25/4$ for $g = -4$, $\omega = e^{i\pi/4}\sqrt{10}$
(b) Nonperiodic orbits $E = 25/4i$ for $g = -4i$, $\omega = \sqrt{10}$

Assuming: $Q(u) = (u-A)(u-B)(u-C)(u-D)$
Two free parameters in solution:

$$u(\zeta) = \frac{B(A-D) + A(D-B)\,\mathrm{sn}\left[\frac{\sqrt{\lambda(B-C)(A-D)}}{2}(\zeta-\zeta_0)\middle|\frac{(A-C)(B-D)}{(B-C)(A-D)}\right]^2}{A-D+(D-B)\,\mathrm{sn}\left[\frac{\sqrt{\lambda(B-C)(A-D)}}{2}(\zeta-\zeta_0)\middle|\frac{(A-C)(B-D)}{(B-C)(A-D)}\right]^2}$$

Reduction:

$$x(t) = A\,\mathrm{sn}\left[(t+t_0)A\sqrt{2E_x}\middle| -\frac{A^4 g}{4E_x}\right]$$

Square root singularity $\Rightarrow$ no linearisation, alternatively
[A.G. Anderson, C. Bender, U. Morone, arXiv:1102.4822]

$$x(t) = x(t + n\omega_1 + m\omega_2) \qquad \text{for } \; n, m \in \mathbb{Z},$$

$$\omega_1 = \frac{4\sqrt{2}}{\sqrt{gA^2+2\omega^2}} K\left[\frac{-A^2 g}{gA^2+2\omega^2}\right] \quad \omega_2 = \frac{i2\sqrt{2}}{\sqrt{gA^2+2\omega^2}} K\left[\frac{2A^2 g + 2\omega^2}{gA^2+2\omega^2}\right]$$

$$n\,\mathrm{Im}\,\omega_1 + m\,\mathrm{Im}\,\omega_2 = 0$$

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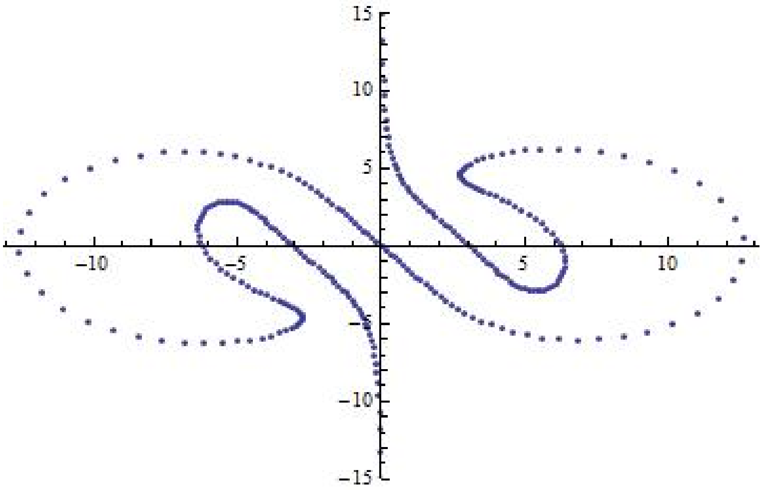
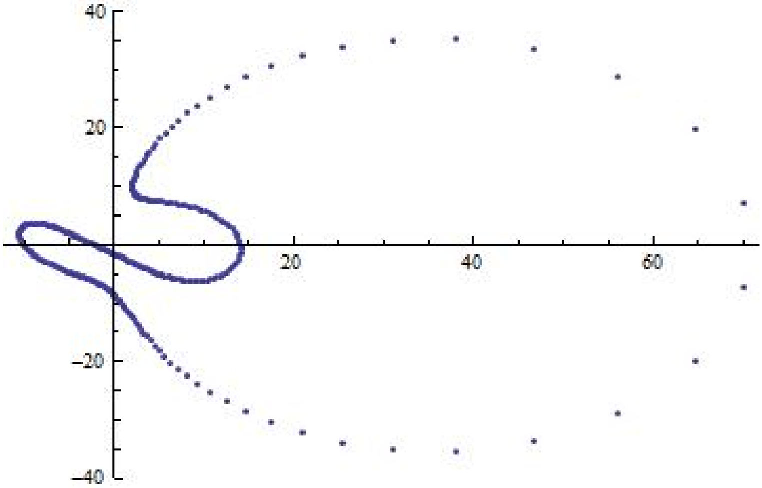
$$n\,\mathrm{Im}\,\omega_1 + m\,\mathrm{Im}\,\omega_2 = 0$$

Note:

One needs $t \to t + it_0$, $t_0 \in \mathbb{R}$ to avoid pole $t = (n\omega_1 + m\omega_2)/2$



The quartic harmonic oscillator from complex modified KDV



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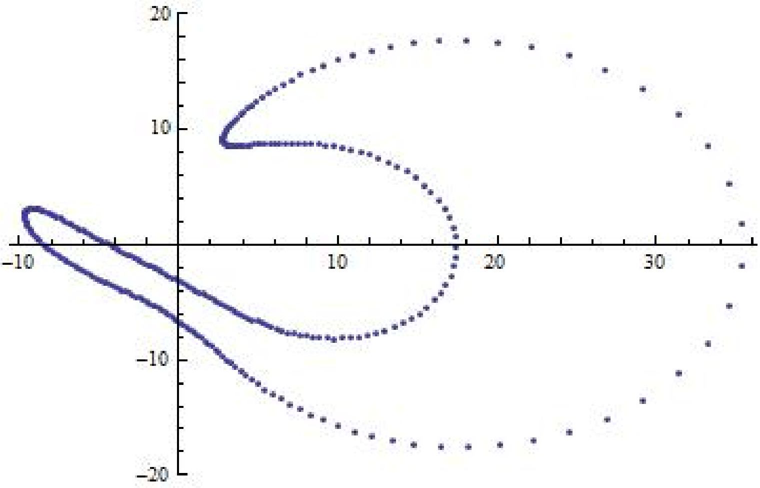

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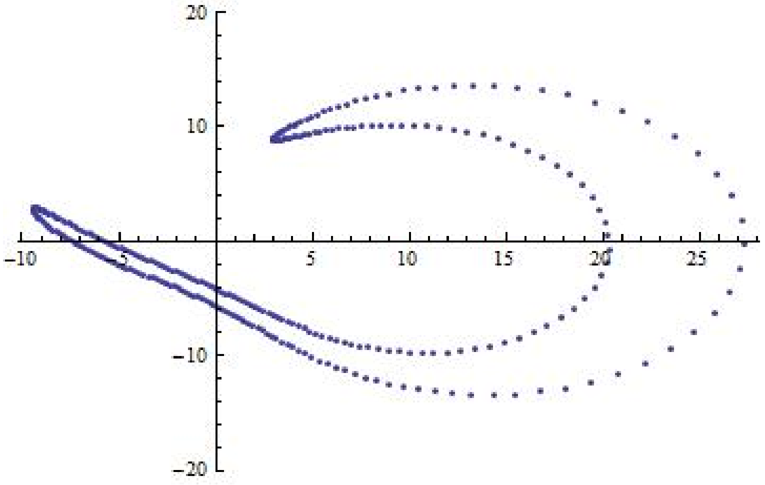

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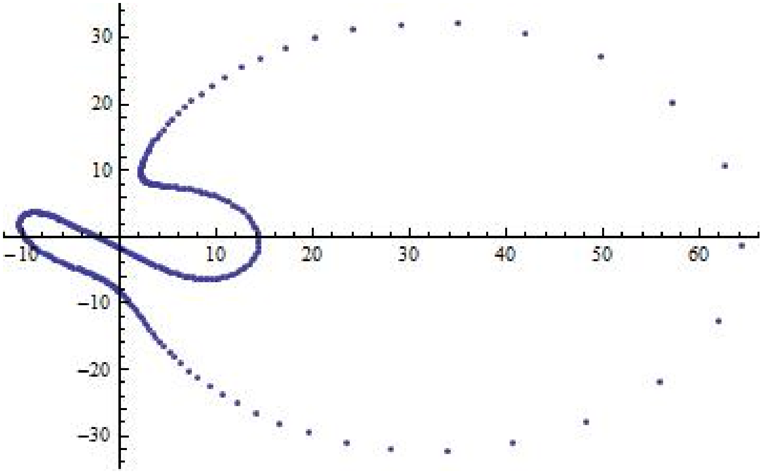

Ito type systems and its deformations

Coupled nonlinear system

$$\begin{aligned} u_t + \alpha v v_x + \beta u u_x + \gamma u_{xxx} &= 0, \qquad \alpha, \beta, \gamma \in \mathbb{C}, \\ v_t + \delta (uv)_x + \phi v_{xxx} &= 0, \qquad \delta, \phi \in \mathbb{C} \end{aligned}$$

Hamiltonian for $\delta = \alpha$

$$\mathcal{H}_I = -\frac{\alpha}{2} u v^2 - \frac{\beta}{6} u^3 + \frac{\gamma}{2} u_x^2 + \frac{\phi}{2} v_x^2$$

$\mathcal{PT}$-symmetries:

$\mathcal{PT}_{++} : x \mapsto -x, t \mapsto -t, i \mapsto -i, u \mapsto u, v \mapsto v$ for $\alpha, \beta, \gamma, \phi \in \mathbb{R}$

$\mathcal{PT}_{+-} : x \mapsto -x, t \mapsto -t, i \mapsto -i, u \mapsto u, v \mapsto -v$ for $\alpha, \beta, \gamma, \phi \in \mathbb{R}$

$\mathcal{PT}_{-+} : x \mapsto -x, t \mapsto -t, i \mapsto -i, u \mapsto -u, v \mapsto v$ for $i\alpha, i\beta, \gamma, \phi \in \mathbb{R}$

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Ito type systems and its deformations

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Deformed models

$$\mathcal{H}^{++}_{\varepsilon,\mu} = -\frac{\alpha}{2}uv^2 - \frac{\beta}{6}u^3 - \frac{\gamma}{1+\varepsilon}(iu_x)^{\varepsilon+1} - \frac{\phi}{1+\mu}(iv_x)^{\mu+1}$$

$$\mathcal{H}^{+-}_{\varepsilon,\mu} = \frac{\alpha}{1+\mu}u(iv)^{\mu+1} - \frac{\beta}{6}u^3 - \frac{\gamma}{1+\varepsilon}(iu_x)^{\varepsilon+1} + \frac{\phi}{2}v_x^2$$

$$\mathcal{H}^{-+}_{\varepsilon,\mu} = -\frac{\alpha}{2}uv^2 - \frac{i\beta}{(1+\varepsilon)(2+\varepsilon)}(iu)^{2+\varepsilon} + \frac{\gamma}{2}u_x^2 - \frac{\phi}{1+\mu}(iv_x)^{\mu+1}$$

$$\mathcal{H}^{--}_{\varepsilon,\mu} = \frac{\alpha}{1+\mu}u(iv)^{\mu+1} - \frac{i\beta}{(1+\varepsilon)(2+\varepsilon)}(iu)^{2+\varepsilon} + \frac{\gamma}{2}u_x^2 + \frac{\phi}{2}v_x^2$$

with equations of motion

$$u_t + \alpha v v_x + \beta u u_x + \gamma u_{xxx,\varepsilon} = 0, \quad u_t + \alpha v_\mu v_x + \beta u u_x + \gamma u_{xxx,\varepsilon} = 0,$$
$$v_t + \alpha(uv)_x + \phi v_{xxx,\mu} = 0, \quad v_t + \alpha(uv_\mu)_x + \phi v_{xxx} = 0,$$

$$u_t + \alpha v v_x + \beta u_\varepsilon u_x + \gamma u_{xxx} = 0, \quad u_t + \alpha v_\mu v_x + \beta u_\varepsilon u_x + \gamma u_{xxx} = 0,$$
$$v_t + \alpha(uv)_x + \phi v_{xxx,\mu} = 0, \quad v_t + \alpha(uv_\mu)_x + \phi v_{xxx} = 0.$$

Solution procedure

- similar as for KdV, but the degrees of the polynomials is higher
- type II $R(v) = v(v-A)^2(v-B)^2$
- eigenvales of the Jacobian:

$$
\begin{aligned}
j_k &= \pm\sqrt{r_A r_\lambda}\left[\cos\left(\frac{3\theta_A}{2}+\frac{\theta_\lambda}{2}\right) r_A - \cos\left(\frac{\theta_A}{2}+\theta_B+\frac{\theta_\lambda}{2}\right) r_B\right] \\
&\quad + i(-1)^k\sqrt{r_A r_\lambda}\left[\sin\left(\frac{3\theta_A}{2}+\frac{\theta_\lambda}{2}\right) r_A - \sin\left(\frac{\theta_A}{2}+\theta_B+\frac{\theta_\lambda}{2}\right) r_B\right]
\end{aligned}
$$

- energy:

$$
\begin{aligned}
E_{T_A} &= \oint_\Gamma \mathcal{H}[v(\zeta)]\frac{dv}{v_\zeta} = \oint_\Gamma \frac{\mathcal{H}[v]}{\sqrt{\lambda}\sqrt{v}(v-A)(v-B)}dv \\
&= -\pi\frac{\sqrt{-\gamma\kappa_2}}{\alpha\sqrt{A}(A-B)}\left[cA^2+\kappa_2 A+\frac{\beta}{3}\left(\frac{c}{\alpha}+\frac{\kappa_2}{\alpha A}\right)^3\right]
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&\quad + i(-1)^k\sqrt{r_A r_\lambda}\left[\sin\left(\frac{3\theta_A}{2}+\frac{\theta_\lambda}{2}\right) r_A - \sin\left(\frac{\theta_A}{2}+\theta_B+\frac{\theta_\lambda}{2}\right) r_B\right]
\end{aligned}
$$

- energy:

$$
\begin{aligned}
E_{T_A} &= \oint_\Gamma \mathcal{H}[v(\zeta)]\frac{dv}{v_\zeta} = \oint_\Gamma \frac{\mathcal{H}[v]}{\sqrt{\lambda}\sqrt{v}(v-A)(v-B)}dv \\
&= -\pi\frac{\sqrt{-\gamma\kappa_2}}{\alpha\sqrt{A}(A-B)}\left[cA^2+\kappa_2 A+\frac{\beta}{3}\left(\frac{c}{\alpha}+\frac{\kappa_2}{\alpha A}\right)^3\right]
\end{aligned}
$$

Solution procedure

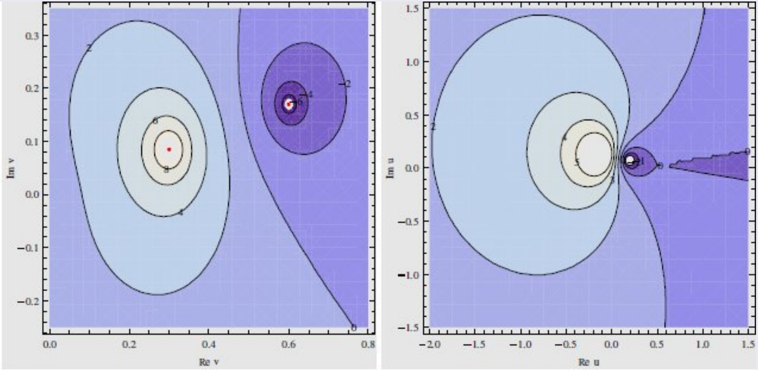
- similar as for KdV, but the degrees of the polynomials is higher
- type II $R(v) = v(v-A)^2(v-B)^2$
- eigenvales of the Jacobian:

$$\begin{aligned} j_k &= \pm\sqrt{r_A r_\lambda}\left[\cos\left(\frac{3\theta_A}{2}+\frac{\theta_\lambda}{2}\right) r_A - \cos\left(\frac{\theta_A}{2}+\theta_B+\frac{\theta_\lambda}{2}\right) r_B\right] \\ & \quad + i(-1)^k\sqrt{r_A r_\lambda}\left[\sin\left(\frac{3\theta_A}{2}+\frac{\theta_\lambda}{2}\right) r_A - \sin\left(\frac{\theta_A}{2}+\theta_B+\frac{\theta_\lambda}{2}\right) r_B\right] \end{aligned}$$

- energy:

$$\begin{aligned} E_{T_A} &= \oint_\Gamma \mathcal{H}\left[v(\zeta)\right]\frac{dv}{v_\zeta} = \oint_\Gamma \frac{\mathcal{H}\left[v\right]}{\sqrt{\lambda}\sqrt{v}(v-A)(v-B)} dv \\ &= -\pi\frac{\sqrt{-\gamma\kappa_2}}{\alpha\sqrt{A}(A-B)}\left[cA^2+\kappa_2 A+\frac{\beta}{3}\left(\frac{c}{\alpha}+\frac{\kappa_2}{\alpha A}\right)^3\right] \end{aligned}$$

Periodic trajectories for type II broken $\mathcal{PT}$-symmetry



$E_{T_A} \approx -0.4275$
(a) v-field
(b) u-field

Conclusions:

- the type of trajectory does not tell which scenario we are in
- all types of fixed points occur (except saddle points)
- there is no chaos by Poincaré-Bendixson theorem
- not Hamiltonian in $Re(u)$, $Im(u)$
- energies can be computed effectively in complex models
- possible to have broken $\mathcal{PT}$-symmetry with real energies
- solitons as in real case, broken $\mathcal{PT}$-symmetry $\Rightarrow$ breather
- deformed models extend over several Riemann sheets
- new features in Ito systems, such as kink or cusp solutions
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