# Analysis of exceptional points in open quantum systems and QPT analogy for the appearance of the resonance state

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# Outline

Orientation:

- Description of open quantum systems (introduce prototype model)
- Exceptional points: basic concepts

Prototype model – semi-infinite chain with endpoint impurity:

- System geometry and Hamiltonian
- Eigenvalue spectrum and system discriminant
- Study of eigenvalues in the vicinity of real-valued EPs – adiabatic transport – eigenvalue expansion

Generalization for open quantum systems:

- Detailed formalism: OQS and EPs (Kato)
- General method to locate and analyze EPs in open quantum systems

QPT analogy for appearance of Fano resonance:

- Dynamical phase transition for channel-coupled resonances
- QPT analogy at EP – spontaneous time-symmetry breaking – complex Helmholtz free energy

# Open Quantum Systems

Open quantum system consists of:

- Discrete system $H_D$
- Embedded in a larger system (continuum) $H_C$
- Coupled via $H_{DC}$

Examples:

- Atoms/molecules interacting with **E** and/or **B** field(s)

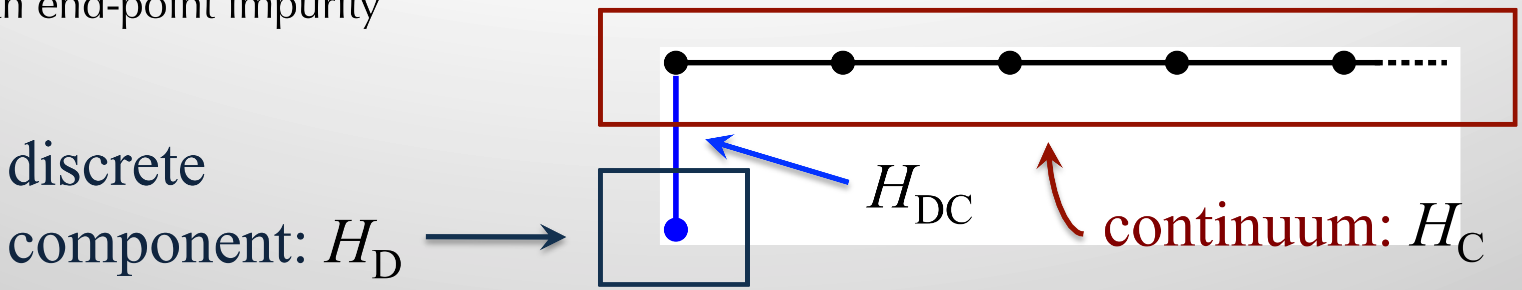
[ $H_2^+$ ion exposed to laser light:
R. Lefebvre, O. Atabek, M. Šindelka, and N. Moiseyev
Phys. Rev. Lett. **103**, 123003 (2009).]

- Antenna leads used to probe an electromagnetic cavity

[ Probing an electromagnetic cavity:
C. Dembowski, et al.
Phys. Rev. Lett. **103**, 123003 (2009).]

# Open Quantum Systems: Prototype model

Prototype: semi-infinite chain
with end-point impurity




$$H = H_D + H_C + H_{DC}$$

$$H = \varepsilon_d d^+ d - \frac{1}{2}\sum_{i=1}^{\infty}(c_i^+ c_{i+1} + c_{i+1}^+ c_i) + \frac{g}{\sqrt{2}}(c_1^+ d + d^+ c_1)$$

# Exceptional Points – Basic Concepts

Exceptional points (EPs):

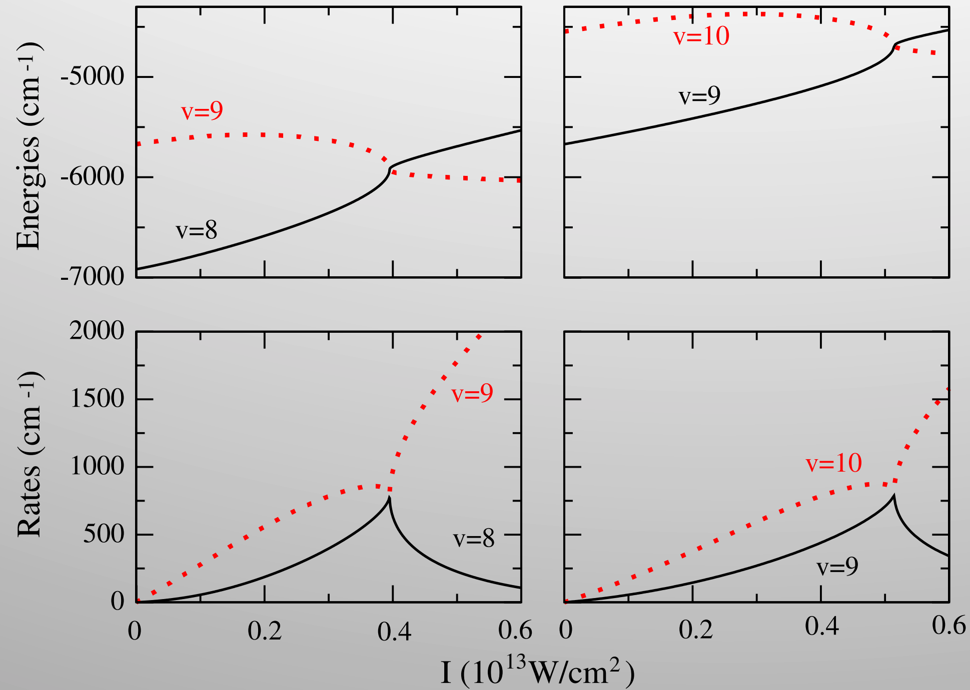
- Occur in the discrete spectrum of a finite Hamiltonian
- 'Defective' points in parameter space at which at least two eigenvalues coalesce
- Eigenvalues share a common branch point in parameter space
- There exist *N*(*N*-1) EPs for an *N*-dimensional system

Adiabatic encirlement of EP: eigenvalues will be rotated into one another

Formalism: Tosio Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin (1980), pp. 62-66.

# EPs in the photodissociation spectrum of $H_2^+$

Lefebvre, et al numerically study EPs in the spectrum of an $H_2^+$ ion exposed to laser light




Adiabatic variation of two system parameters:

$$I = I_{max} \sin(\phi/2),$$

$$\lambda = \lambda_0 + \delta\lambda \sin(\phi)$$

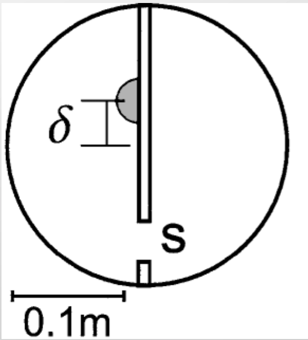
Encircling the EP we may populate the ν=9 state through the ν=8 state, etc.


R. Lefebvre, O. Atabek, M. Šindelka, and N. Moiseyev
Phys. Rev. Lett. **103**, 123003 (2009).

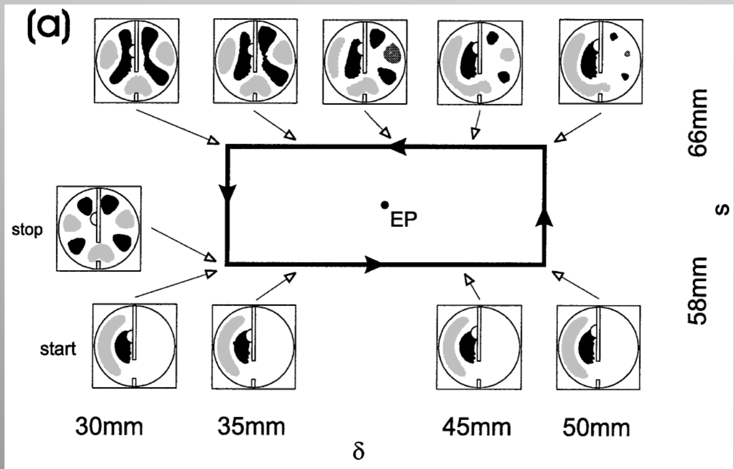
# EPs: Experimental observation in the modes of a microwave cavity

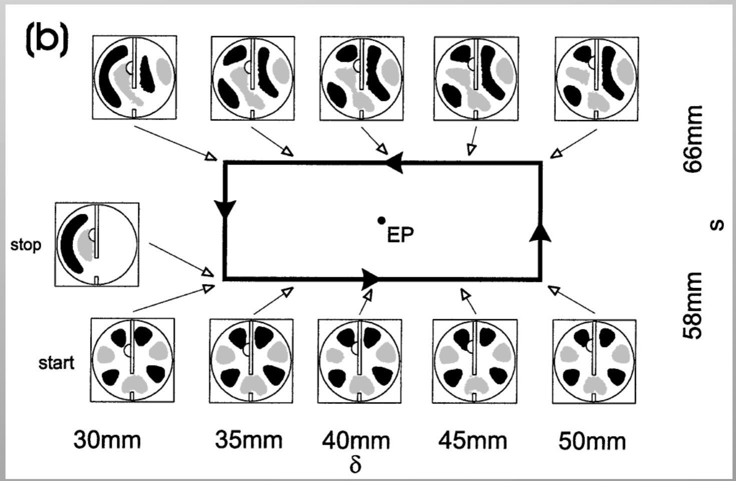
C. Dembowski, H.-D. Gräf, H. L. Harney, A. Heine, W. D. Heiss, H. Rehfeld, and A. Richter, Phys. Rev. Lett. **86**, 787 (2001).

Top view of microwave cavity



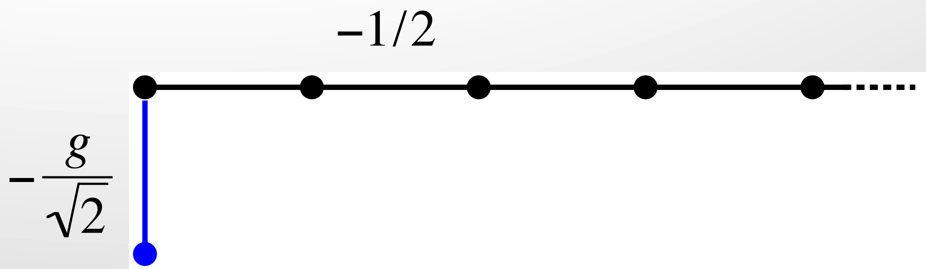

Adiabatic variation of the 8th and 9th eigenmodes:



# Prototype model: Hamiltonian

System geometry:




$$H = \varepsilon_d d^+ d - \frac{1}{2}\sum_{i=1}^{\infty}(c_i^+ c_{i+1} + c_{i+1}^+ c_i) + \frac{g}{\sqrt{2}}(c_1^+ d + d^+ c_1)$$

(gives rise to a **quadratic** discrete spectrum)

# Prototype model: band spectrum

Introduce Fourier series:

$$c_k^\dagger = \sqrt{\frac{2}{\pi}} \sum_{i=1}^{N} \sin nk \, c_n^\dagger$$

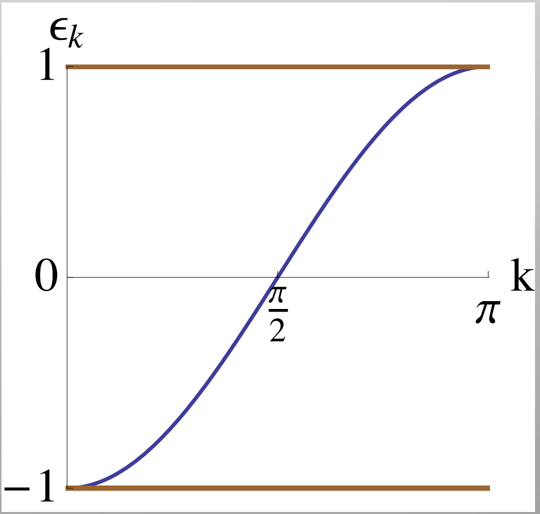
to re-write Hamiltonian as

$$H = \varepsilon_d d^\dagger d + \int_0^\pi \varepsilon_k c_k^\dagger c_k + g \int_0^\pi V_k (c_k^\dagger d + d^\dagger c_k)$$

$$(N \to \infty)$$

Continuum:

$k \in [0, \pi]$ on $\varepsilon_k = -\cos k$



# Prototype model: discrete spectrum

$$H = \varepsilon_d d^\dagger d + \int_0^\pi \varepsilon_k c_k^\dagger c_k + g\int_0^\pi V_k (c_k^\dagger d + d^\dagger c_k)$$

Final diagonalization may be performed. Obtain the discrete spectrum from:

$$\left\langle d \left| \frac{1}{z-H} \right| d \right\rangle = \frac{1}{z - \varepsilon_d - \Xi_s(z)} \quad \text{in which} \quad \Sigma(z) = g^2 (z - \sqrt{z^2 - 1})$$

We have:

$$\boxed{z - \varepsilon_d = \Sigma(z) = g^2 (z - \sqrt{z^2 - 1})}$$

# Prototype model: system discriminant I

Square the dispersion relation $z - \varepsilon_d = g^2(z - \sqrt{z^2 - 1})$

to obtain quadratic polynomial equation with $\boxed{q(z) = 0}$

$$q(z) = (1 - 2g^2)z^2 - 2\varepsilon_d(1 - g^2)z + \varepsilon_d^2 + g^4$$

Combine with

$$q'(z) = 2(1 - 2g^2)z - 2\varepsilon_d(1 - g^2) = 0$$

To obtain discriminant:

$$D(\varepsilon_d, g) = -4g^4\left(1 - 2g^2\right)\left(\varepsilon_d^2 - (1 - 2g^2)\right)$$

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$f_1(g) \equiv 1 - 2g^2$ $\qquad$ $f_2(\varepsilon_d, g) \equiv \varepsilon_d^2 - (1 - 2g^2)$

# Prototype model: system discriminant II

Discriminant:

$$D(\varepsilon_d, g) = -4g^4 f_1(g) f_2(\varepsilon_d, g) \qquad \begin{cases} f_1(g) = 1 - 2g^2 \\ f_2(\varepsilon_d, g) = \varepsilon_d^2 - (1 - 2g^2) \end{cases}$$

$f_2(\varepsilon_d, g) = 0$ gives two EPs: $\varepsilon_d = \bar{\varepsilon}_\pm \equiv \pm\sqrt{1 - 2g^2}$

For $g < 1/\sqrt{2}$ $\left(g > 1/\sqrt{2}\right)$ these EPs are real-valued (pure imaginary).

$g = 1/\sqrt{2}$ ➔ special case.

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Then $f_1(g) = 0$ shows $g = 1/\sqrt{2}$ is an EP in its own right!

(pure coincidence)

We will focus on real-valued case $g < 1/\sqrt{2}$ for now.

# Discrete eigenvalues and adiabatic properties of the EPs

Solve $q(z) = 0$ for exact solutions (quadratic):

$$z_{\pm}(\varepsilon_d, g) = \varepsilon_d \frac{1-g^2}{1-2g^2} \pm g^2 \frac{\sqrt{\varepsilon_d^2 - (1-2g^2)}}{1-2g^2}$$

We have (anti-)resonant state for $\left|\varepsilon_d\right| > \sqrt{1-2g^2} > 0$ with complex part $\Gamma_{\pm} = \pm g^2 \frac{\sqrt{\varepsilon_d^2 - (1-2g^2)}}{1-2g^2}$

Adiabatic coordinate rotation:

$$\left(\varepsilon_d(\theta)\right)^2 = 1 - 2g^2 + \delta e^{i\theta}$$

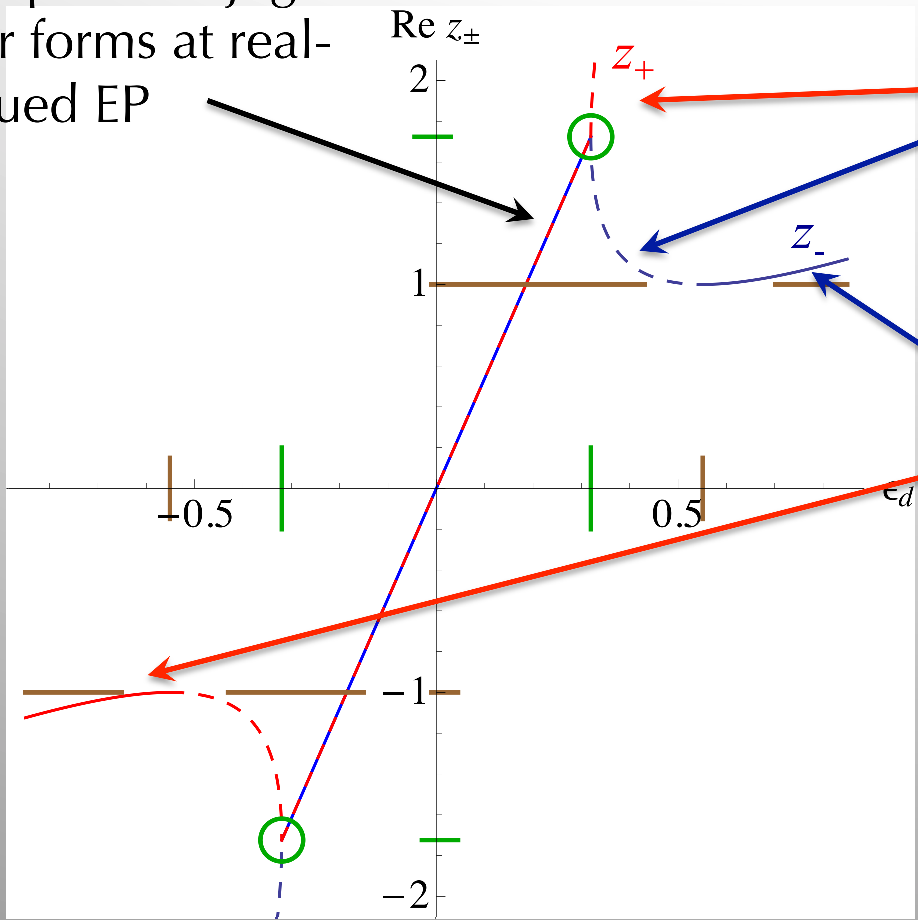
Then $\Gamma_{\pm}(\theta) = \pm \frac{g^2\sqrt{\delta}}{(1-2g^2)} e^{i\theta/2}$

and $\Gamma_{\pm}(2\pi) \rightarrow \Gamma_{\mp}(0)$

# Discrete spectrum: level shift

complex conjugate pair forms at real-valued EP

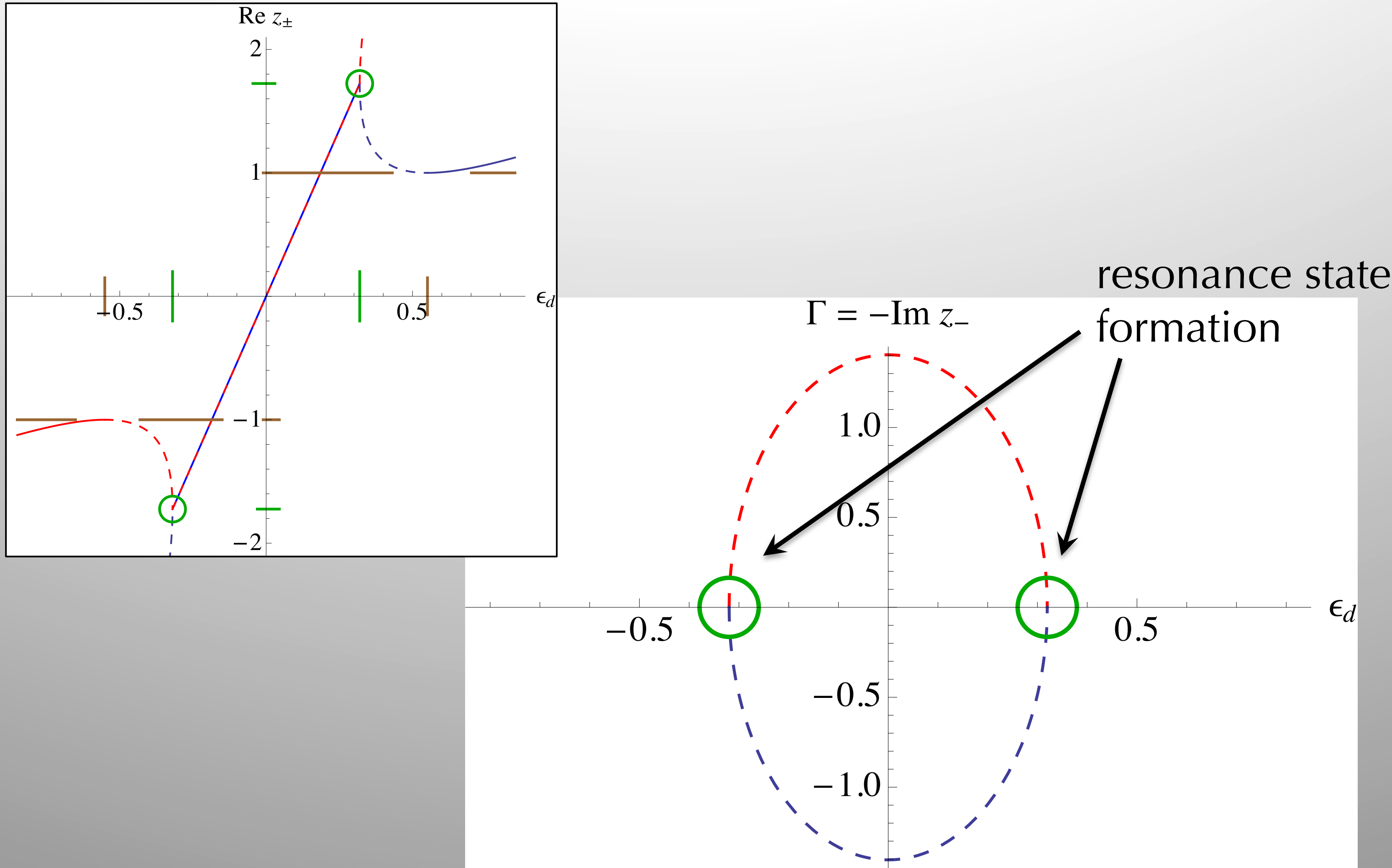
two anti-bound states ($2^{nd}$ sheet) collide at real-valued EP $\varepsilon_d = \bar{\varepsilon}_\pm$



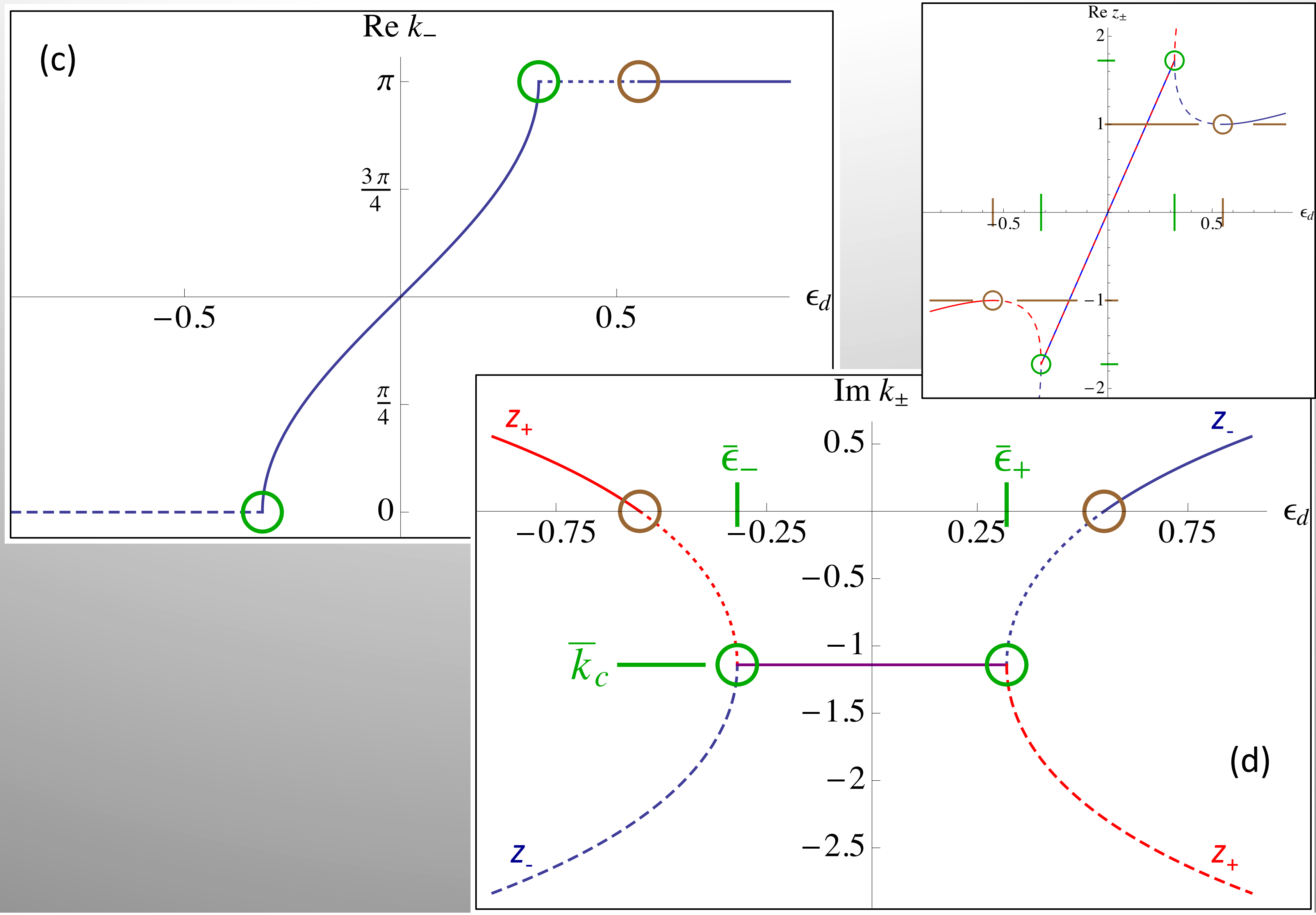

bound states ($1^{st}$ sheet)

$g = .67 < 1/\sqrt{2}$

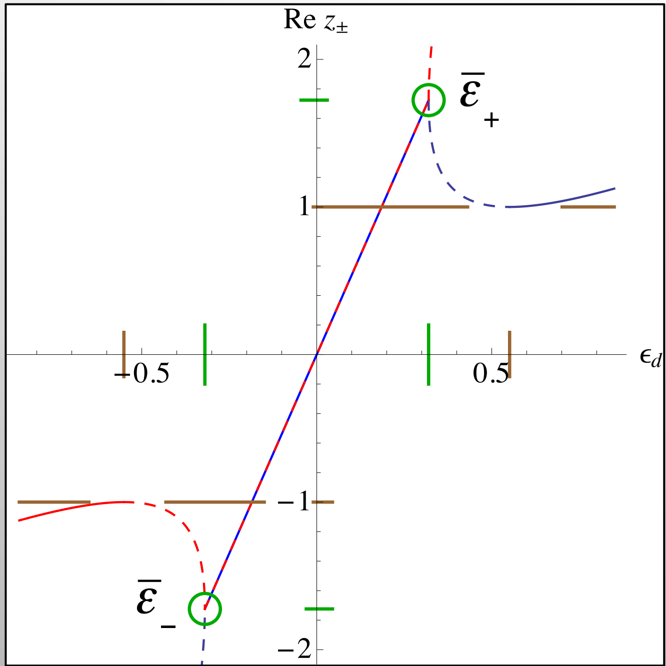
# Discrete spectrum: resonance state at EP

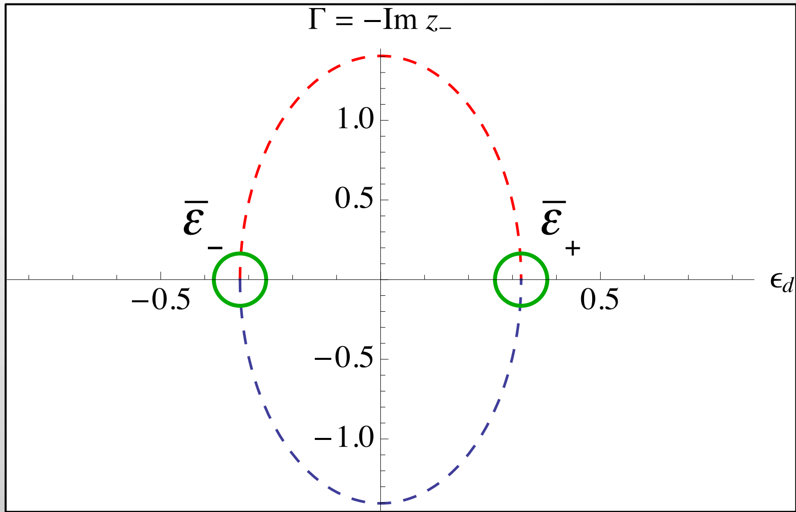


# Discrete spectrum: effective $k$ values



# Discrete spectrum: eigenvalue expansion at EP




$$f_2(\varepsilon_d, g) \equiv \varepsilon_d^2 - (1-2g^2) = (\varepsilon_d - \bar{\varepsilon}_+)(\varepsilon_d - \bar{\varepsilon}_-)$$

Eigenvalue expansion in the vicinity of $\varepsilon_d \approx \bar{\varepsilon}_\pm$

$$z_s = \frac{1+\bar{\varepsilon}_\pm^2}{2\bar{\varepsilon}_\pm} + s\frac{1-\bar{\varepsilon}_\pm^2}{2\bar{\varepsilon}_\pm^2}\left(f_2(\varepsilon_d, g)\right)^{1/2} + \frac{1}{2\bar{\varepsilon}_\pm}\sum_{n=2}^{\infty}\left(\frac{s\left(f_2(\varepsilon_d, g)\right)^{1/2}}{\bar{\varepsilon}_\pm}\right)^n \qquad (s = \pm)$$

# Comparison: superconducting nanowires driven with electric currents

J. Rubinstein, P. Sternberg, and Q. Ma, Phys. Rev. Lett. **99**, 167003 (2007).

*Bifurcation Diagram and Pattern Formation of Phase Slip Centers in Superconducting Wires Driven with Electric Currents*

fractional power series for eigenvalue

complex conjugate pair formation

eigenvalue is 1. To find the behavior of the spectrum near $I_{co}$ we set the current $I$ to be $I = I_{co} + \epsilon a$. Here $\epsilon$ is a small positive number, and we introduce $a$ to determine through its sign the direction in which we move from $I_{co}$. We then consider an expansion of the form

$$\lambda = \mu_0 + \epsilon^{1/2}\mu_1 + \epsilon\mu_2 + \ldots,$$
$$u = u_0 + \epsilon^{1/2}u_1 + \epsilon u_2 + \ldots. \quad (4)$$

The nonanalytic nature of the expansion for $\lambda$ is a consequence of the Jordan form of the spectral problem at the critical value $I = I_{co}$. The leading order term in (4) is

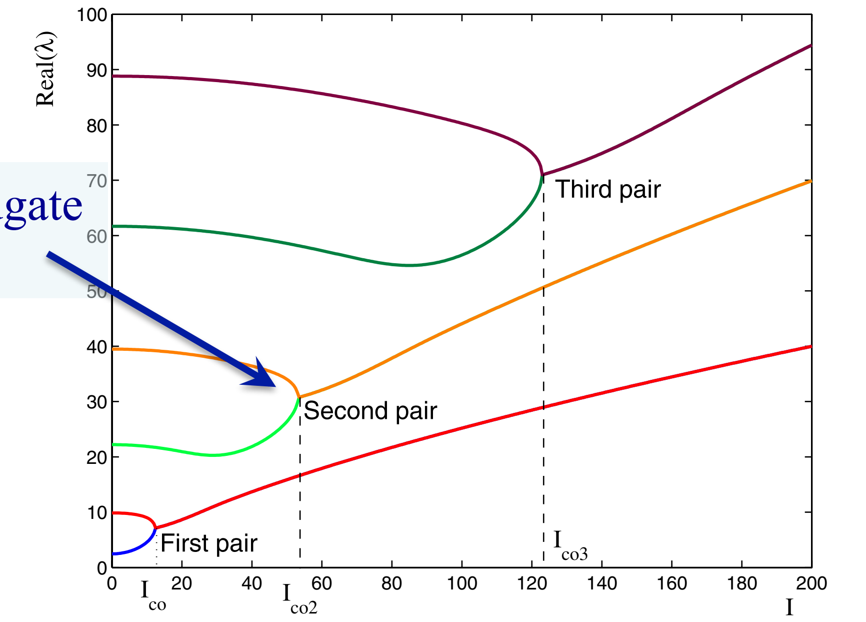



FIG. 1 (color online). The real parts of the first six eigenvalues of the $PT$-symmetric spectral problem (3).

# Generalization of discussion: Open quantum systems – Formalism

Generic Hamiltonian: $H = H_D + H_C + H_{DC}$

Specific case, single level discrete sector:

$$H_D = \varepsilon_d d^+ d \qquad H_{DC} = gV \qquad (\text{system parameters}: \ \varepsilon_d, g)$$

Finite level spectrum given by the roots of:

$$z - \varepsilon_d = \Sigma(z)$$

From this equation we obtain the system discriminant:

$$D(\varepsilon_d, g) = f_1(\varepsilon_d, g) \times f_2(\varepsilon_d, g) \times \cdots \times f_r(\varepsilon_d, g)$$

$f_i(\overline{\varepsilon}_d, \overline{g}) = 0$ ← necessary and sufficient condition for an EP $\boxed{\overline{\varepsilon}_d = \overline{\varepsilon}_d(\overline{g})}$

# Formalism continued:
# Cycle structure around an EP

Eigenvalues organize into *cycles* in the vicinity of a given EP:

$$\{z_1(x),\ \ldots,\ z_{p-1}(x),\ z_p(x)\},\ \{z_{p+1}(x),\ \ldots,\ z_{p+q-1}(x),\ z_{p+q}(x)\},\ \ldots$$

Adiabatic revolution around exceptional point $x = \bar{x}$

$$\{z_1(x),\ \ldots,\ z_{p-1}(x),\ z_p(x)\} \rightarrow \{z_2(x),\ \ldots,\ z_p(x),\ z_1(x)\}$$

For a given cycle, the *center* is the energy value where the $p$ eigenvalues coalesce:

$$\bar{z}_c = z_1(x = \bar{x}) = \ldots = z_p(x = \bar{x})$$

# Formalism continued: Eigenvalue expansion

We may expand the eigenvalues in the vicinity of the EP:

$$z_h(x) = \bar{z}_c + \beta_1 \omega^h (x - \bar{x})^{1/p} + \beta_2 \omega^{2h} (x - \bar{x})^{2/p} + \ldots \qquad \text{(Kato)}$$

with $\omega = e^{2\pi i/p}$, $h = 0, \ \ldots, \ 1 - p$

Heuristic generalization:

$$z_h(x) = \bar{z}_c + \beta_1 \omega^h (f_j(x))^{1/p} + \beta_2 \omega^{2h} (f_j(x))^{2/p} + \ldots$$

$f_j(x)$ are polynomials from the system discriminant

Tosio Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin (1980), pp. 62-66.

# General method to find EPs in open quantum systems

Idea: take advantage that *eigenvalue derivative diverges at the EP*

From general dispersion $z - \varepsilon_d = \Sigma(z)$ (1)

take derivative to obtain

$$\frac{d\Sigma(z_0)}{dz_0} = 1 - \frac{1}{\partial z_0 / \partial \varepsilon_d} \qquad (2)$$

at the EP :

$$\left.\frac{d\Sigma(z_0)}{dz_0}\right|_{z_0 = \bar{z}_c} = 1 \qquad (3)$$

Use (3) to find the *center* $\bar{z}_c$ then plug this result in (1) to locate EP.

Expand (2) to obtain higher terms and use

$$p = \frac{1}{2\pi i} \oint_{C_{\bar{z}_c}} \frac{\partial}{\partial z} \log \left\langle d \left| \frac{1}{z - H} \right| d \right\rangle dz$$

# Motivation for *p*-value equation

$$p = \frac{1}{2\pi i}\oint_{C_{\bar{z}_c}} \frac{\partial}{\partial z}\log\left\langle d\left|\frac{1}{z-H}\right|d\right\rangle dz = \frac{1}{2\pi i}\oint_{C_{\bar{z}_c}} \frac{\frac{d}{dz}[z-\varepsilon_d-\Xi(z)]}{z-\varepsilon_d-\Xi(z)}dz$$

In the vicinity of the EP:

$$z-\varepsilon_d-\Xi(z) \sim (z-\bar{z}_c)^p$$

So we have

$$\frac{\frac{d}{dz}[z-\varepsilon_d-\Xi(z)]}{z-\varepsilon_d-\Xi(z)} \sim \frac{p}{z-\bar{z}_c}$$

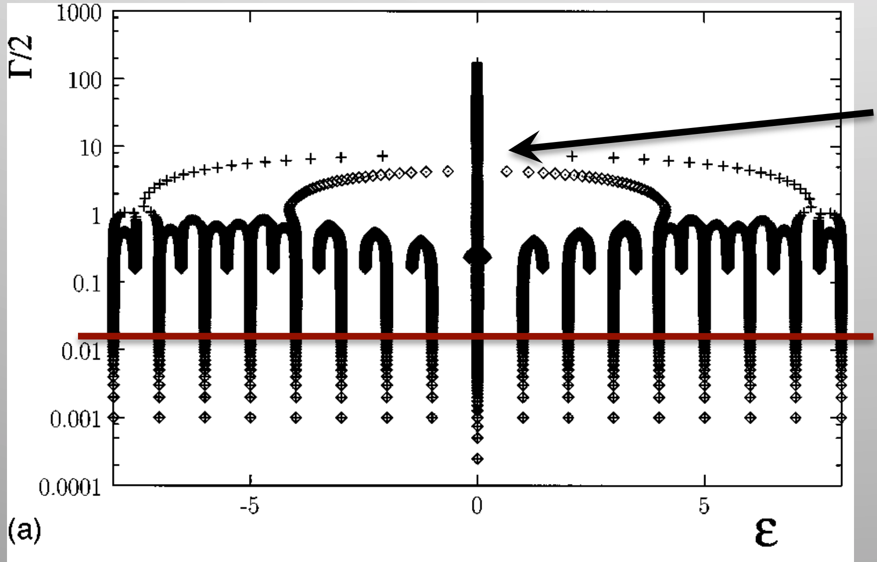
# Dynamical phase transition in open quantum systems

C. Jung, M. Müller, and I. Rotter, Phys. Rev. E 60, 114 (1999).

They study a line of resonant states coupled through a common decay channel

parameterized through the common coupling parameter $\alpha$

Complex energy plane




Above $\alpha_{crit}$: one resonance 'takes off,' aligns with decay channel; system re-organization

Small $\alpha$: resonances have a similar shape/magnitude

➔ Interprets *decay rate as the order parameter* in this dynamical phase transition

# QPT analogy for the real-valued EPs

For our case, we propose:

Order parameter $\Gamma_{res} \sim \lambda^{1/2}$

With $\lambda = f_2(\varepsilon_d, g) = \varepsilon^2 - \bar{\varepsilon}_\gamma^2$

Correlations through the resonant state:

$$C_{res}(x;d) = \frac{1}{2\pi i} \oint_{\Gamma_-} dz \left\langle x \left| \frac{1}{z-H} \right| d \right\rangle \sim e^{\log\sqrt{1-2g^2}\,x} e^{i\varphi_{res} x}$$

With $\varphi_{res} = \arctan\left(\lambda^{1/2} / \varepsilon_d\right)$

'channel correlation' $\xi \sim \lambda^{-1/2}$

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Scaling: $\Gamma_{res} \sim \xi^{-z} \sim \lambda^{1/2}$

Dynamical exponent: $z = 1$

(time dimension)

# Conclusions

- Generalization of Kato's expression for eigenvalue expansion in the vicinity of EPs
- General technique to determine the position of EPs and the eigenvalue expansion in open quantum systems
- We applied this to our prototype model – semi-infinite chain with endpoint impurity
- Transition from real-valued EPs to complex EPs:
  - Fano resonance to avoided level crossing
  - Behavior of system across critical value $g_{cr}$
- QPT analogy at real-valued EPs:
  - Decay rate as order parameter
  - Channel correlations and dynamical critical exponent
- Complex free energy – fractional powers as critical exponents