Non-Dirac-hermitian Supersymmetric Quantum Systems: A Deconstruction

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References

- arXiv:1105.2495
- JPA **44** (2011) 215307 [arXiv:1101.2120]
- JPA **43** (2010) 125203 [arXiv:0908.1321]
- JPA **38** (2005) 7313 [quant-ph/0501087]

Plan of the seminar

- The role of metric
- Methods to find the metric
- Outline of a deconstruction
- Example: non-SUSY model
- Clifford algebra: A non-Dirac hermitian realization
- Non-Dirac hermitian supersymmetric theory
  - Jaynes-Cummings Hamiltonian
  - Many-particle systems
  - Pseudo-hermitian Superconformal Symmetry
- Summary of Results

The role of metric

- A consistent quantum description of a class of non-dissipative non-Dirac-hermitian Hamiltonian is admissible
  - Entirely real spectra
  - Unitarity
  - Completeness of eigenstates

- The class is characterized by an unbroken combined $\mathcal{PT}$ symmetry or pseudo-hermiticity of the Hamiltonian

- A modified norm and/or a positive-definite metric in the Hilbert space is required for consistency and completeness

- Explicit form of the metric or knowledge of the modified norm is not known for majority of the models

  Description of this class of non-dissipative non-Dirac-hermitian Hamiltonian is thus incomplete

Methods to find the metric η_+

- Spectral Decomposition
  - Prior knowledge of a complete set of states
  - Involves non-trivial summation/integration
  - Very often, approximations are made

- Perturbation Theory
  - The description is not exact
  - Effective hermitian Hamiltonian or metric is known only up to a order of the perturbation

- Moyal Product
  - Very limited applications
  - The description is not exact

- Group Theory

  Limited applications to quantum systems with dynamical symmetry

Consequently, not many non-Dirac-hermitian quantum system with EXACT metric is known

Systems with a pre-determined metric

What about constructing Pseudo-hermitian quantum system with a pre-determined metric?

Motivations:

- Pseudo-hermitian deformation of known Dirac-hermitian quantum systems
  - Stable bound states for enlarged parameter space
  - Systematic study of allowed metric for a given system
  - Isospectral Hamiltonian
- Examples of more exactly solved non-dissipative non-Dirac-hermitian quantum systems with a complete description
  - Check validity of approximate and/or numerical methods
  - Check other relevant ideas to the field

A Method for Deconstruction

- Consider a Hilbert space $\mathcal{H}_{\eta_+^b}$ that is endowed with a Positive-definite metric η_+^b

  Example: $\eta_+^b := e^{-\delta L_3},\ L_3 = x_1 p_2 - x_2 p_1,\ \delta \in R$

- Find a realization of the basic CCR in terms of operators which are hermitian in $\mathcal{H}_{\eta_+^b}$

  Example:

$$X_1 = x_1\ coshw + i x_2\ sinhw,$$

$$X_2 = -i x_1\ sinhw + x_2\ coshw, \quad X_3 = x_3$$

$$P_1 = p_1\ coshw + i p_2\ sinhw,$$

$$P_2 = -i p_1\ sinhw + p_2\ coshw,\ P_3 = p_3$$

$$w \equiv \delta + i\xi, \quad \xi\ \in\ R$$

- Hamiltonian constructed out of X_i, P_i's are hermitian in $H_{\eta_+^b}$, but, not necessarily in $\mathcal{H}_D$

- Several examples with non-central potentials are allowed

An Example

A non-Dirac-hermitian Hamiltonian:

$$\begin{aligned} H &= \frac{p^2}{2m} - \frac{e^2}{r} + e\mathcal{E}X_1 \\ &= \frac{p^2}{2m} - \frac{e^2}{r} + e\mathcal{E}\left(x_1\ coshw + i\ x_2\ sinhw\right), \quad \mathcal{E} \in R \end{aligned}$$

Each operator B_i in $\mathcal{H}_D$ is related to the corresponding operator A_i in $\mathcal{H}_{\eta_+^b}$:

$$A_i = (\rho^b)^{-1} B_i \rho^b, \quad \rho^b := \sqrt{\eta_+^b}$$

Example : $X = (\rho^b)^{-1} x \rho^b$, $H = (\rho^b)^{-1} h \rho^b$

$$h = \frac{p^2}{2m} - \frac{e^2}{r} + e\mathcal{E}x_1$$

Eigenstates & eigenvalues:

$$\psi_H = (\rho^b)^{-1}\psi_h, \quad \langle\langle A\rangle\rangle_{\eta_+^b} = \langle B\rangle_{\mathcal{H}_D}$$

Example: Integrable Spin-chain models

A metric

$$\zeta_+ := \prod_{i=1}^{N} e^{-2\delta_i \mathcal{S}_i^z}$$

Pseudo-hermitian spin-chain model:

$$\begin{aligned} H_A &= \sum_{i=1}^{N-1} [\Gamma \left(e^{w_i - w_{i+1}} \mathcal{S}_i^+ \mathcal{S}_{i+1}^- + e^{-(w_i - w_{i+1})} \mathcal{S}_i^- \mathcal{S}_{i+1}^+ \right) \\ &+ \Delta \mathcal{S}_i^z \mathcal{S}_{i+1}^z + (A_i coshw_i - iB_i sinhw_i) \mathcal{S}_i^x \\ &+ (B_i coshw_i + iA_i sinhw_i) \mathcal{S}_i^y + C_i \mathcal{S}_i^z] \end{aligned}$$

Isospectral partner:

$$\begin{aligned} h &= \sum_{i=1}^{N-1} [\Gamma \left(\mathcal{S}_i^x \mathcal{S}_{i+1}^x + \mathcal{S}_i^y \mathcal{S}_{i+1}^y \right) \\ &+ \Delta \mathcal{S}_i^z \mathcal{S}_{i+1}^z + A_i \mathcal{S}_i^x + B_i \mathcal{S}_i^y + C_i \mathcal{S}_i^z] \end{aligned}$$

- Several integrable limits [T Deguchi and PKG, JPA **42**, 475208(2009)]

- Pseudo-hermitian deformations of Dicke model, quadratic form of boson(fermion) operators, Calogero-type models have been studied

Clifford Algebra(CA) and a metric

Real Clifford Algebra:

$$\{\xi_p, \xi_q\} = 2\delta_{pq}, \quad p, q = 1, 2, \ldots, 2N$$

Generators of $O(2N)$ group:

$$J_{pq} := \frac{i}{4} [\xi_p, \xi_q]$$

A positive-definite metric with N real parameters γ_i:

$$\eta_+^f := \prod_{i=1}^{N} e^{2\gamma_i J_{iN+i} - \gamma_i}, \quad \rho_f := \sqrt{\eta_+^f}$$

- Positivity may be shown by expressing ξ_p in terms of fermionic variables

$$\psi_i := \frac{1}{2} (\xi_i - i\xi_{N+i}), \quad \psi_i^\dagger := \frac{1}{2} (\xi_i + i\xi_{N+i})$$

- A more general metric with $N(2N-1)$ number of real parameters is admissible

CA : A pseudo-hermitian realization

General Prescription:

$$\Gamma_p := (\rho^f)^{-1} \xi_p \rho^f$$

A particular realization:

$$\begin{aligned} \Gamma_i &= \xi_i \; cosh\gamma_i + i\xi_{N+i} \; sinh\gamma_i, \\ \Gamma_{N+i} &= -i\xi_i \; sinh\gamma_i + \xi_{N+i} \; cosh\gamma_i \end{aligned}$$

- Γ_p's are non-Dirac-hermitian and admit entirely real eigenvalues. Γ_p's are hermitian in $\mathcal{H}_{\eta_+^f}$
- Known matrix representations of ξ_p lead to matrix representations of Γ_p
- Γ_p's may be used to construct $2^N \times 2^N$ pseudo-hermitian and pseudo-unitary matrices depending on $2^{2N} + N$ real parameters

Complexification of real CA

Fermionic operators:

$$\Psi_i := \frac{1}{2}(\Gamma_i - i\Gamma_{N+i}) = e^{-\gamma_i}\psi_i$$
$$\Psi_i^\dagger := \frac{1}{2}(\Gamma_i + i\Gamma_{N+i}) = e^{\gamma_i}\psi_i^\dagger$$

The algebra:

$$\{\Psi_i, \Psi_j\} = 0 = \{\Psi_i^\dagger, \Psi_j^\dagger\}, \quad \{\Psi_i, \Psi_j^\dagger\} = \delta_{ij}$$

- Ψ_i's may be interpreted as 'fermionic operators' ψ_i with purely imaginary gauge coupling
- Fock space representation of the states are identical for $\psi_i, \psi_i^\dagger$ and $\Psi_i, \Psi_i^\dagger$
- Eigenstates of the total fermion number operators in $\mathcal{H}_D$ and $\mathcal{H}_{\eta_+^f}$ differ by a phase:

$$|f_1, \ldots, f_N\rangle_{\mathcal{H}_{\eta_+^f}} = \prod_{k=1}^{N} e^{\gamma_k f_k} |f_1, \ldots, f_N\rangle_{\mathcal{H}_D}$$

$f_i = 0, 1 \ \forall \ i$

Supersymmetry: A brief review

Superalgebra:

$$H = \{Q, Q^\dagger\}, \quad Q^2 = 0 = {Q^\dagger}^2,$$
$$[H, Q] = 0 = [H, Q^\dagger]$$

By construction,

- H is hermitian $\Longrightarrow$ real eigenvalues E_n
- H is semi-positive definite $\Longrightarrow$ $E_n \geq 0$

A particular realization:

$$Q = \sigma_+ a, \quad Q^\dagger = \sigma_- a^\dagger, \quad \sigma_\pm := \sigma_1 \pm i\sigma_2$$
$$a = p - i\frac{\partial W}{\partial x}, \quad a^\dagger = p + i\frac{\partial W}{\partial x},$$

$W \equiv W(x)$ superpotential

$\sigma_{1,2,3}$ are Pauli Matrices

Exact solvability for Shape Invariant potentials

SUSY & JC

Superoscillators

$$W = \frac{1}{2}x^2 \Rightarrow [a, a^\dagger] = 1$$

The Jaynes-Cummings Hamiltonian:

$$\begin{aligned} H &= a^\dagger a + \frac{1}{2}[a, a^\dagger](\sigma_3 + 1) \\ &+ \lambda \left(e^{i\theta}\sigma_+ a + e^{-i\theta}\sigma_- a^\dagger\right) \\ &= \{Q, Q^\dagger\} + \lambda \left(e^{i\theta} Q + e^{-i\theta} Q^\dagger\right) \\ & \quad \lambda \in R, \quad 0 \leq \theta \leq 2\pi \end{aligned}$$

- Exactly solvable
- SUSY can be realized for the Dicke model also
- Relevance of Dicke model: Optics, QPT, RMT,Chaos, spintronics

A non-Dirac-hermitian JC

- Consider Clifford Algebra with $N = 1$:

$$\xi_1 := \sigma_1, \xi_2 := \sigma_2, \quad \gamma_5 := \sigma_3$$

- Replace $\xi_{1,2} \to \Gamma_{1,2}, \quad \sigma_\pm \to e^{\pm\gamma}\sigma_\pm$

The non-Dirac-hermitian Hamiltonian

$$\begin{aligned} H &= a^\dagger a + \frac{1}{2}[a, a^\dagger](\sigma_3 + 1) \\ &+ \lambda \left(e^{\gamma + i\theta}\sigma_+ a + e^{-(\gamma + i\theta)}\sigma_- a^\dagger \right) \end{aligned}$$

- Entirely real spectra, unitarity,
  metric $\eta_+ := I \otimes e^{-\gamma\sigma_3}$

- For Dicke model, non-Dirac-hermitian deformations involving $a, a^\dagger$ produces more general examples

- QPT[Deguchi and PKG, PRE**80**, 021107(2009)], RMT[T. Deguchi, PKG and K. Kudo, PRE**80**, 026213(2009)]

Pseudo-hermitian Supersymmetry

The metric

$$\eta_+ := \eta_+^b \otimes \eta_+^f$$

- Use Γ_p or $\Psi_i, \Psi_i^\dagger$ instead of $\xi_p, \psi_i, \psi_i^\dagger$
- Use X_i, P_i instead of x_i, p_i

SUSY Hamiltonian

$$\tilde{H} = \frac{1}{2}\sum_{i=1}^{N}\left(p_i^2 + W_i^2 - W_{ii}\right) + \sum_{i,j=1}^{N} e^{\gamma_i - \gamma_j} W_{ij}\psi_i^\dagger\psi_j$$

$W \equiv W(X)$ is superpotential, $W_i = \frac{\partial W}{\partial X_i}$ etc.

- $\tilde{H}$ is not hermitian in the sense of Dirac: $\tilde{H}^\dagger \neq \tilde{H}$
- $\tilde{H}$ is pseudo-hermitian: $\tilde{H}^\dagger = \eta_+ \tilde{H} \eta_+^{-1}$
- $\tilde{H}$ is isospectral with a Dirac-hermitian H:

$$H = \rho \tilde{H} \rho^{-1}, \quad \rho := \sqrt{\eta_+}$$

Pseudo-hermitian Superconformal Symmetry

Choice of the superpotential:

$$W = -\lambda \, lnr, \quad r = \left(\sum_{i=1}^{N} x_i^2 \right)^{\frac{1}{2}}$$

leads to a pseudo-hermitian superconformal Hamiltonian.

- Pseudo-hermitian realization of $OSp(2|2)$ superalgebra

- Ground state is not even plane-wave normalizable. Neither in $\mathcal{H}_D$ nor in $\mathcal{H}_{\eta_+}$

- A compact operator constructed out of the pseudo-hermitian generators of $O(2,1)$ bosonic sub-algebra of $OSp(2|2)$ can be used to study the unitary time-evolution of the system

- Extended $SU(1,1|2)$ superconformal symmetry with pseudo-hermitian generators for $N = 2$

Summary

- Description of pseudo-hermitian quantum systems is incomplete without the knowledge of the positive-definite metric η_+ in the Hilbert space.

  expectation values/correlation functions ×

- Finding η_+ is a highly non-trivial task for systems with infinite dimensional Hilbert space.

- A method to construct pseudo-hermitian quantum systems with a pre-determined η_+ has been described

  1. Choose η_+ and introduce $\mathcal{H}_{\eta_+}$ with $\langle\langle\cdot|\cdot\rangle\rangle_{\eta_+}$
  2. Find a realization of the operators which are non-Dirac-hermitian, but, are hermitian in $\mathcal{H}_{\eta_+}$
  3. Construct diagonalizable operators in $\mathcal{H}_{\eta_+}$

- Method is applicable to non-relativistic, relativistic and supersymmetric quantum systems

- Pseudo-hermitian quantum systems thus constructed are necessarily isospectral with Dirac-hermitian systems

- Examples include pseudo-hermitian deformations of spin-chain models, Dicke model, quadratic form of N boson(fermi) operators, super-conformal quantum systems, Calogero-type models etc.

- These models may serve as prototype examples for testing different ideas related to the subjects, including validity of any perturbative and/or numerical computation.

- Experimental realizations of some of these systems are desirable

GRAFFITI

Unwritten first law of Physics:

Anything that is not forbidden is compulsory

THANK YOU