PT-symmetric Sinusoidal Optical Lattices at the Symmetry-Breaking Threshold

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Outline

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   The Optical Potential $V = V_0(\cos(2\pi x/a) + i\lambda \sin(2\pi x/a))$

2. Spectrum

3. Dynamics for $\lambda < 1$

4. **Special Case $\lambda = 1$**
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1. PT-symmetry and Optics

Start with scalar Helmholtz equation

$$\left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial z^2} + k^2\right) E(x,z) = 0$$

Thinking of a medium with $n = n_0(1 + v(x))$, where $v \ll 1$.

Write $E = e^{ik_0 z}\psi(x,z)$. Eq^n becomes

$$\left(\frac{\partial^2}{\partial z^2} + 2ik_0\frac{\partial}{\partial z} + \frac{\partial^2}{\partial x^2} + (k^2 - k_0^2)\right)\psi = 0$$

Paraxial approximation:

Neglect first term, assuming that $\partial^2\psi/\partial z^2 \ll \partial^2\psi/\partial x^2$
Also $k^2 - k_0^2 \approx 2k_0^2 v$

So get analogue Schrödinger eq$^{\text{n}}$

$$i\frac{\partial\psi}{\partial z} + \frac{1}{2k}\frac{\partial^2\psi}{\partial x^2} + k\,v(x)\psi = 0$$

with z playing role of t. Excess refractive index $n_0 v(x)$ can be complex, and can be made PT symmetric.

Such optical systems have all kinds of interesting properties[†].

[†] R. El-Ganainy et al., Optics Letters **32**, 2632 (2007).

The Sinusoidal Potential

Consider Schrödinger eq^{n}

$$i\frac{\partial\psi}{\partial z} = -\left(\frac{\partial^2}{\partial x^2} + V(x)\right)\psi,$$

with $V = V_0(\cos(2\pi x/a) + i\lambda\sin(2\pi x/a))$

Look for "energy" eigenf^{ns} of form $\psi(x,z) \propto e^{-i\beta z}$. Eigenfunctions are Bloch functions with periodicity condition

$$\psi_k(x+a) = e^{ika}\psi_k(x)$$

In first instance $k = k(\beta)$, which have to invert to get spectrum $\beta = \beta(k)$

2. Spectrum

Turns out that spectrum is completely real (PT-symmetry unbroken) for $\lambda \leq 1$, and becomes partly complex for $\lambda > 1$.

Can be understood in terms of **equivalent Hermitian Hamiltonian** h, induced by similarity transformation

$$h = e^{-\frac{1}{2}Q} H \; e^{\frac{1}{2}Q}$$

In general Q very difficult to find, but in this case

$$Q = \theta \widehat{p} \equiv -i\theta d/dx$$

All it does† is shift x to $x + \frac{1}{2} i\theta$.

†C. M. Bender, H. F. Jones and R. J. Rivers
Phys. Lett. **B 625**, 333 (2005)

(i) $\lambda < 1$

$$H = p^2 - V_0(\cos 2x + i\lambda \sin 2x)$$

$(a = \pi)$ can be converted into the Hermitian Hamiltonian

$$h = p^2 - V_0\sqrt{(1-\lambda^2)}\cos 2x$$

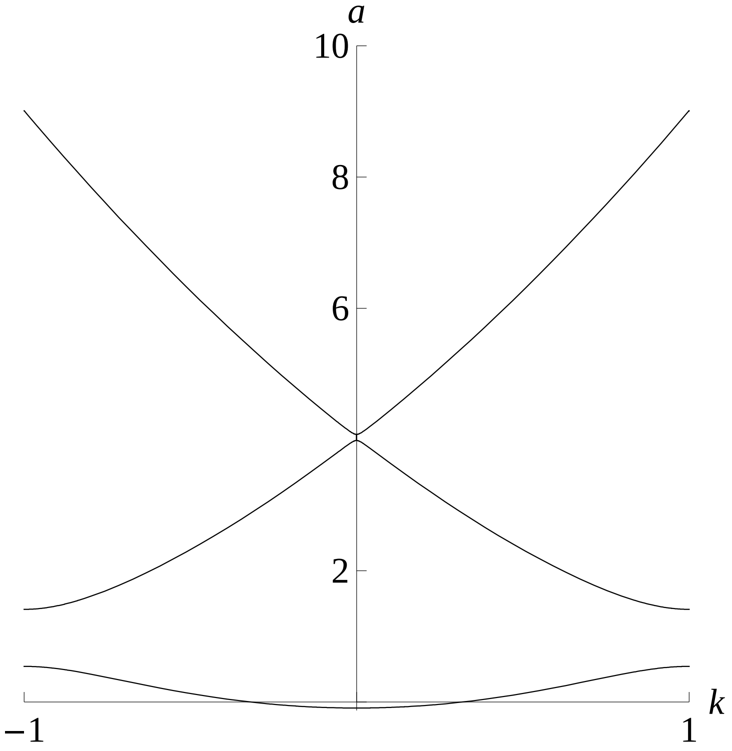
by the complex shift $x \to x + \frac{1}{2}i\theta$, where $\lambda = \tanh\theta$, provided that $\lambda < 1$.

Then Schrödinger eq^{n} for $\psi(x,z)$ becomes the **Mathieu** eq^{n} for $\varphi(x,z) = \psi(x + \frac{1}{2}i\theta)$:

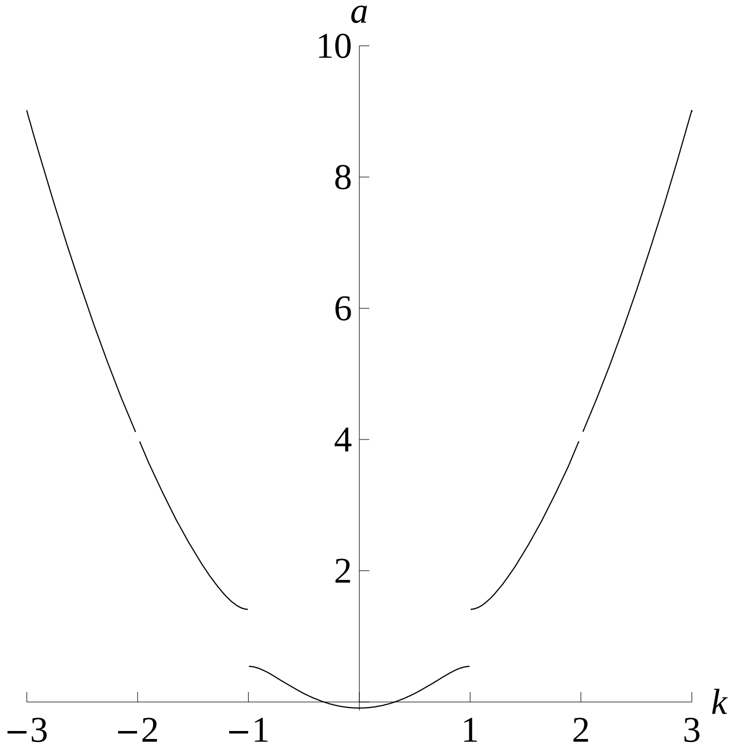
$$\varphi'' + (a - 2q\cos 2x)\varphi = 0,$$

with $q = -\sqrt{(1-\lambda^2)}$ and $a = \beta$.

Spectrum can actually be found by use of built-in Mathematica functions MathieuA and MathieuB, and looks like the following ($V_0 = 2$, $\lambda = 0.9$):



Reduced zone scheme



Extended zone scheme

(ii) $\lambda > 1$

In this case, analogue Schrödinger equation for h is

$$\varphi'' + (a + iV_0\sqrt{(}\lambda^2 - 1)\sin 2x)\varphi = 0,$$

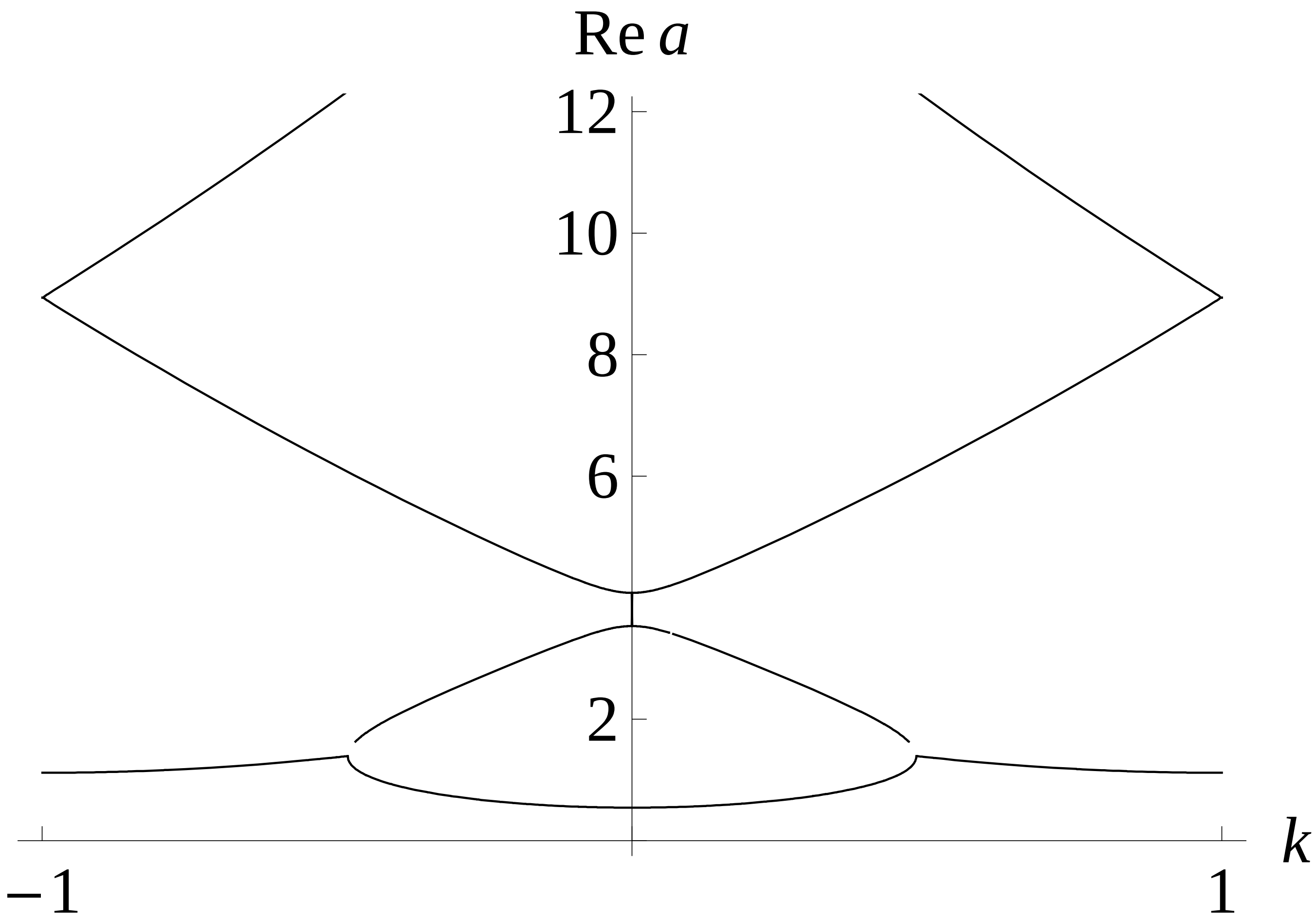
No longer Hermitian, and has some complex eigenvalues† .

That means exponential growth/decay in z .

By shift $x \to x - \pi/2$, this again becomes the Mathieu equation, but with q pure imaginary.

The functions MathieuA and MathieuB are still defined in Mathematica and can again be used to map out the band structure:

†All complex above $V_0 \approx 0.888437$
N. Midya, B. Roy and R. Choudhury, Phys. Lett. A **374** (2010) 2605



Band structure (real part) for $\lambda = 1.4$ in reduced zone scheme.

(iii) $\lambda = 1$

In this case we just have

$$h = p^2$$

So spectrum is just $\beta = k^2$.

However, similarity transformation is now singular since $\theta \to \infty$, so can't be used to transform wave-functions.

This case needs to be treated separately.

3. Dynamics for $\lambda < 1$

Can generate dynamics (development in z) by **method of stationary states** [and exploiting similarity transformation† between ψ and φ]:

- Expand initial WF $\psi(x, z = 0)$ as a superpos^{n} of orthonormalized "energy" eigenf^{ns} $\psi_{k_r}(x)$:

$$\psi(x, z = 0) = \sum_r c_r \psi_{k_r}(x)$$

- with

$$c_r = \frac{\int \psi^*_{-k_r}(-x)\psi(x, z = 0)dx}{\int \psi^*_{-k_r}(-x)\psi_{k_r(x)}dx}$$

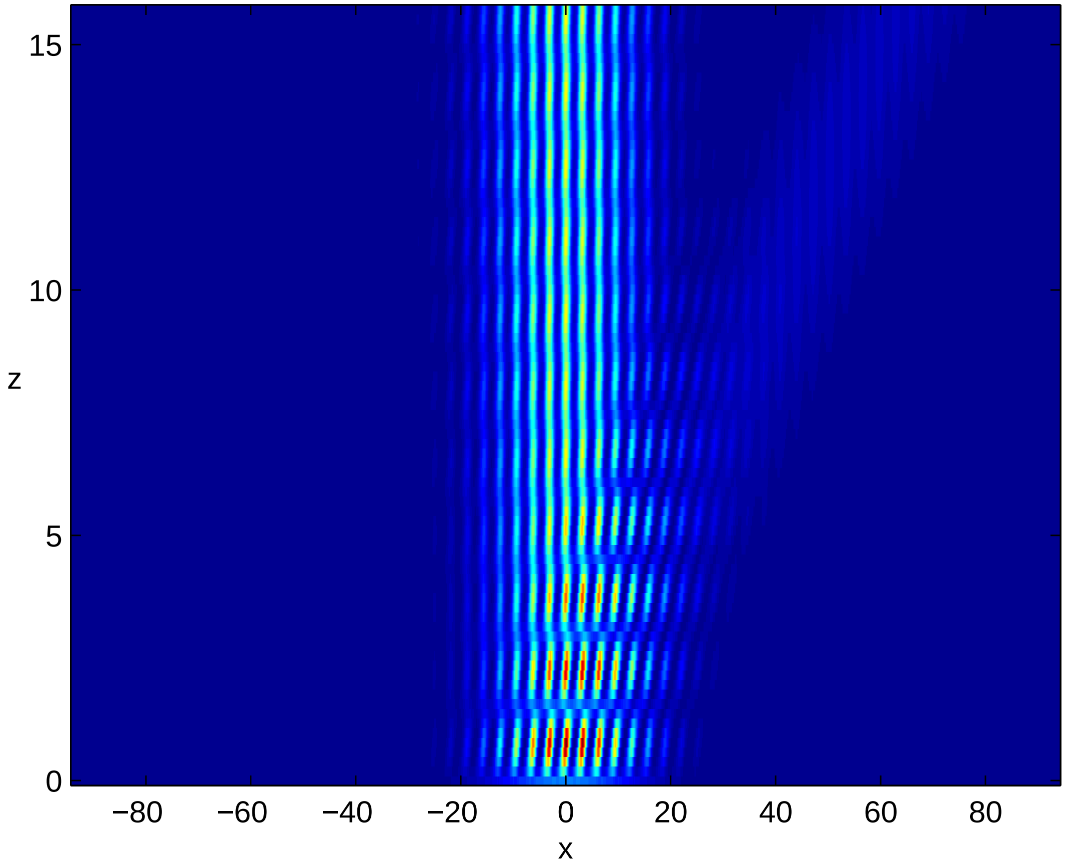
†HFJ: arXiv:1009.5784

- and then

$$\psi(x,z) = \sum_r c_r \psi_{k_r}(x) e^{-i\beta(k_r)z}$$

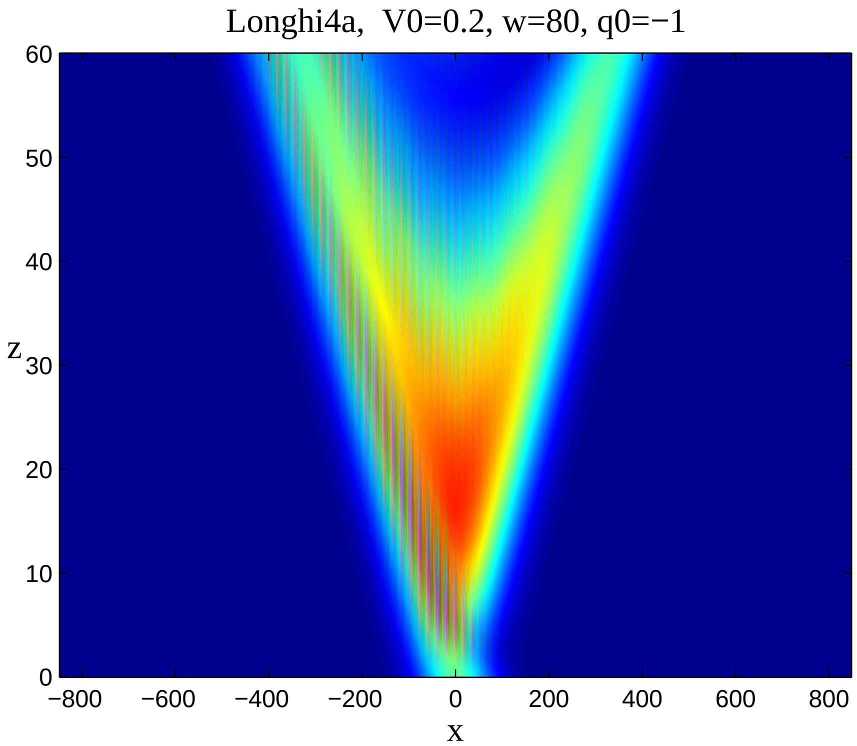
Take a Gaussian wave-packet $\psi(x,0) = e^{-(x/w)^2}$. Then intensity pattern† is

†R. El-Ganainy et al.
Optics Letters **32**, 2632 (2007)



Intensity pattern ($|\psi(x,z)|^2$) for $\lambda = 0.9$, $w = 6\pi$, $V_0 = 2$

Alternatively, with input $\psi(x,0) = e^{-(x/w)^2 + ik_0 x}$ get†



†S. Longhi
Phys. Rev. A **81**, 022102 (2010)

4. Special Case $\lambda = 1$[†]

Already mentioned that can't use similarity transformation to h $\because \theta \to \infty$.

But can still use method of stationary states.

Analogue Schrödinger eq$^{\mathsf{n}}$ is

$$-\frac{d^2\psi}{dx^2} - V_0 \exp(2i\pi x/a)\psi = \beta\psi.$$

[†]E-M Graefe & HFJ
arXiv:1104.2838, $\rightarrow$ Phys. Rev. A

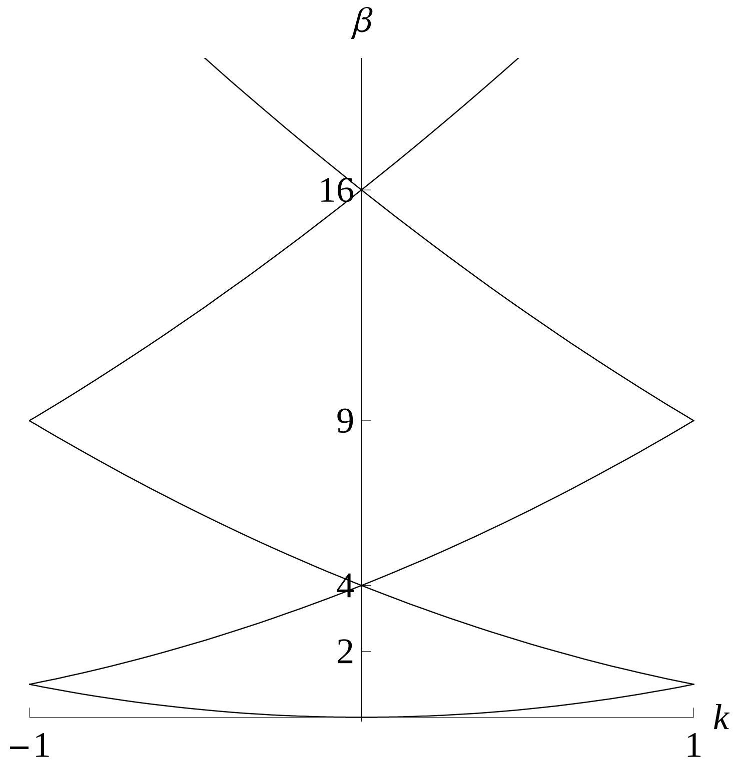
With change of variable $y = y_0 \exp(i\pi x/a)$, where $y_0 = (a/\pi)\sqrt{V_0}$, this becomes the modified Bessel equation

$$y^2 \frac{d^2\psi}{dy^2} + y\frac{d\psi}{dy} - (y^2 + q^2)\psi = 0,$$

with $q^2 = \beta(a/\pi)^2$. Spectrum is free spectrum. Away from BZ boundaries $q \in Z$, Bloch wave-functions are precisely

$$\psi_k(x) = I_q(y)$$

Makes life easier if we choose $a = \pi$.
Then $k = q$ and $y = \sqrt{V_0} e^{ix}$.



β
16
9
4
2
k
−1
1

(i) Jordan associated functions

For k not an integer $I_k(y)$ and $I_{-k}(y)$ are linearly independent.

But when $k = n$ we have $I_{-n}(y) = I_n(y)$.

i.e. $\exists$ only one eigenfunction for $\beta = n^2$. So eigenfunctions do not form a complete set.

Spectral singularity, exceptional point, Jordan block.

Simple matrix example:

$$M = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

has only one eigenvalue λ and one eigenvector $u = (1, 0)$

To complete basis, need additional vector $v = (0, 1)$, which is a sol^{n} of the generalized eigenvalue eq^{n}

$$(M - \lambda)v = u$$

In general eigenvectors are orthogonal in sense

$$(\tilde{u}_L)_1 (u_R)_2 = 0$$

for $\lambda_1 \neq \lambda_2$

At the special point u is self-orthogonal, but $(\tilde{u}_L)(v_R) = 1$.

Essentially same true when M is linear diff^{l} operator H.

Eigenf^{ns} orthogonal w.r. to PT metric

$$\int dx \psi_{-k}(x)\psi_{k'}(x) = \sigma(k)\delta_{kk'}$$

Here

$$\sigma(k) \equiv \int dx \psi_{-k}(x)\psi_{k}(x)$$

is not positive definite, but alternates from band to band. Hence in this degenerate case it has to vanish at BZ boundaries.

In fact it turns out that

$$\int dx \psi_{-k}(x)\psi_{k}(x) = a \ \mathrm{sinc}(ka)$$

Jordan associated function $\varphi_n(x) \equiv \chi_n(y)$ defined as derivative of ψ_n w.r.t. to E, i.e.

$$\varphi_n = \frac{1}{2k} \frac{d\psi_n}{dk}\Big|_{k=n}$$

and satisfies "generalized eigenvalue equation"

$$(H - E_n)\varphi_n = \psi_n$$

$\chi_n(y)$ undefined up to multiples of $I_n(y)$ and $K_n(y)$, and must satisfy Bloch periodicity.

Derivative def$^{\text{n}}$ gives really simple formulae for φ_n in terms of $I_m(y)$.

In particular $\chi_1(y) = -I_0(y)/(2y)$, which automatically has correct periodicity

Time-dependence for Jordan Block

Back to matrix example

If, at $t = 0$,

$$w = au + bv$$

then at a later time t

$$w = e^{-iHt}(au + bv)$$

Now $(H - \lambda)u = 0 \Rightarrow e^{-iHt}u = e^{-i\lambda t}u$

whereas $(H - \lambda)v = u \Rightarrow$

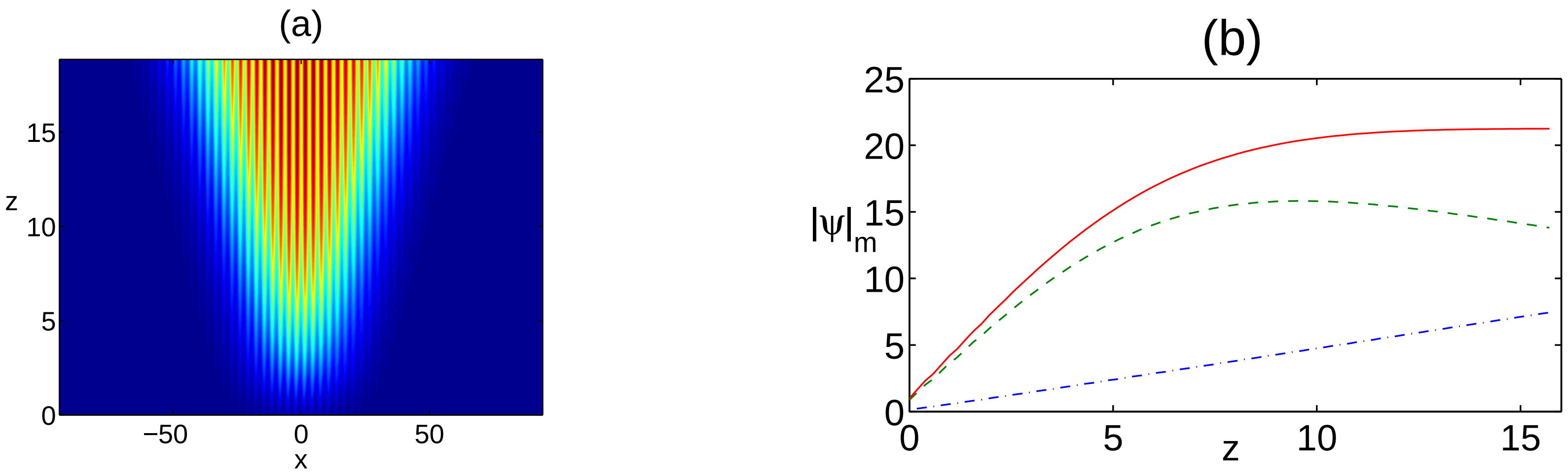
$$
\begin{aligned}
e^{-iHt}v &= e^{-i\lambda t}e^{-i(H-\lambda)t}v \\
&= e^{-i\lambda t}\left[1 - i(H-\lambda)t + \ldots\right] v \\
&= e^{-i\lambda t}(v - itu)
\end{aligned}
$$

So generically, Jordan block $\Rightarrow$ linear t-dependence.

Expect same in our case, where H is a differential operator. BUT, it turns out that this is not the case, and that the linear growth is saturated†

†S. Longhi, Phys. Rev. A **81**, 022102 (2010).

(ii) Saturation $\left(g(x) = e^{-(x/w)^2 + ik_0 x}\right)$



(a) $|\psi(x,z)|$ as a f^n of x, z

(b) Max. value (red line) of $|\psi(x,z)|$ as a f^n of z.

- Blue dashed-dotted line: Jordan block contrib^{ns} only.

- Green dashed line: other contrib^{ns} only.

Parameters are: $V_0 = 2$, $w = 6\pi$ and $k_0 = -1$.

- How can this possibly be??

- How can an explicit z dependence be compensated by other (oscillatory) contributions?

- Answer is that

(a) Other contributions "know" about Jordan term

(b) Cancellation can only occur mathematically for limited range of z

- But that's OK because lattice is **finite** in x-direction

(c) Can show cancellation explicitly

(a) - other contributions “know” about Jordan term

$\psi(x,0) \equiv g(x)$ is expanded as

$$\psi(x,0) = c_0\psi_0(x) + \sum_{k\neq n} c_k\psi_k(x) + \sum_{n>0} [\alpha_n I_n(y) + \beta_n\chi_n(y)]$$

Then $\psi(x,z)$ is given by

$$\begin{aligned}\psi(x,z) &= c_0\psi_0(x) + \sum_{k\neq n} c_k\psi_k(x)e^{-ik^2 z} \\ &\quad + \sum_{n>0} [(\alpha_n - iz\beta_n)I_n(y) + \beta_n\chi_n(y)]e^{-in^2 z}\end{aligned}$$

Now c_k is given by

$$c_k = \frac{\int_0^\pi dx \psi_{-k}(x) G_k(x)}{2N \int_0^\pi dx \psi_{-k}(x) \psi_k(x)},$$

and β_n by

$$\beta_n = \frac{\int_0^\pi dx I_n(y) G_n(x)}{2N \int_0^\pi dx I_n(y) \chi_n(y)}.$$

where

$$G_k(x) \equiv \sum_{m=-N}^{N-1} e^{-i\pi mk} g(x + m\pi).$$

Numerator of β_n is a continuation of that of c_k

Denominator of c_k would vanish at $k = n$, and denominator of β_n is $\propto$ its **derivative** in k

Moreover, numerator $\widehat{c}_k$ of c_k is highly peaked around $k = 1$. Namely

$\widehat{c}_k \propto e^{-\epsilon^2 w^2/4}$, where $\epsilon = k - 1$

while denominator $n_k \propto \epsilon$

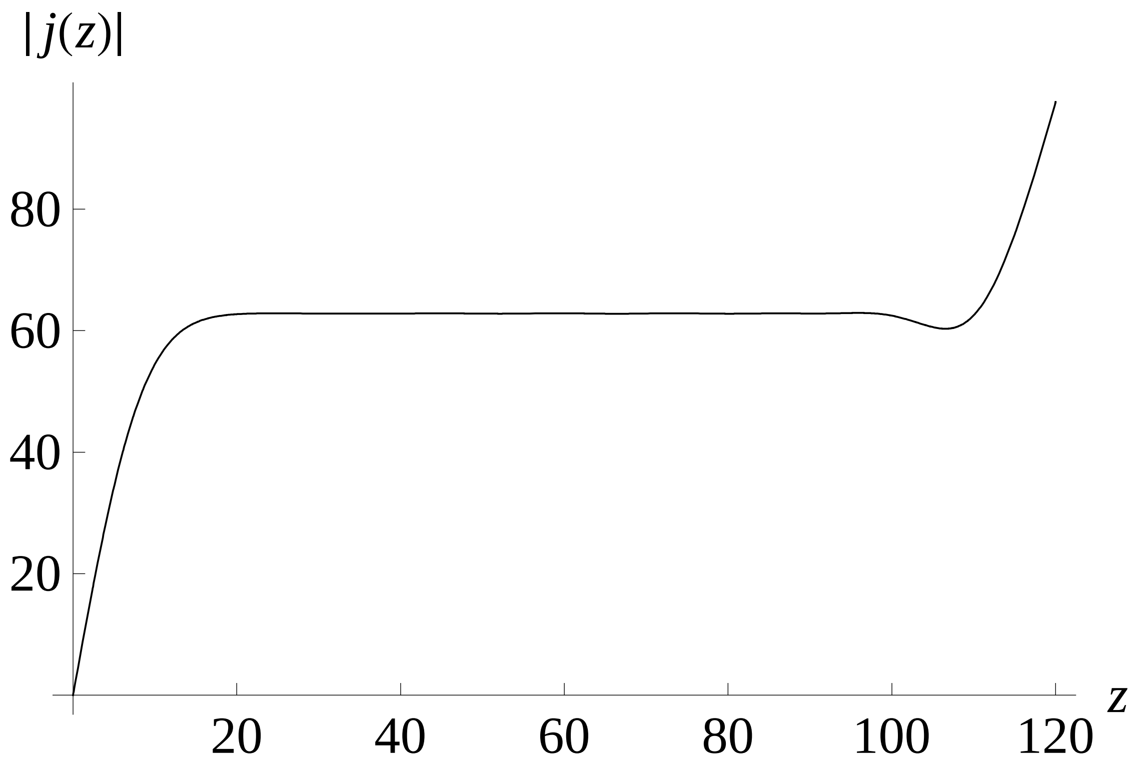
(b) - cancellation can only occur mathematically for limited range of z

So total contribution from neighbourhood of $k = 1$ and $k = -1$ is

$$\text{const} \times I_1(y) e^{-iz} \left[z + \frac{1}{2} \sum_{r=1} \left(\frac{2}{\epsilon_r} \right) \sin(2\epsilon_r z) e^{-i\epsilon_r^2 (z + w^2/4)} \right]$$

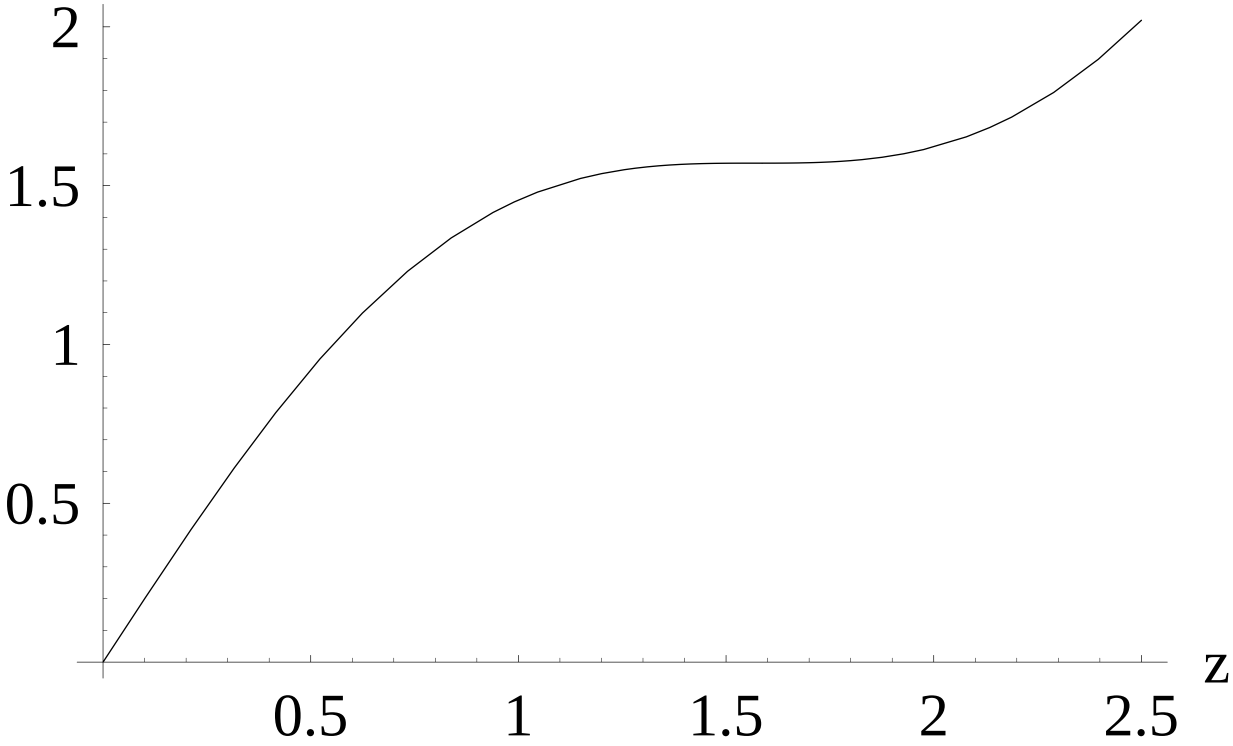
where $\epsilon_r = r/N$

Turns out that this is flat for a very long way:



This is for box $|x| \leq N\pi$, where $N = 40$

Hint comes from function $z + \sin(2z)/2$:



But previous function is **really** flat.

(c) - can show cancellation explicitly

Can understand the extreme flatness of the plateau in terms of Jacobi ϑ_3 functions. Essential function is

$$
\begin{aligned}
j(z) &\equiv z + \sum_{r=1}^{\infty} \left(\frac{1}{\epsilon_r}\right) \sin(2\epsilon_r z) e^{-\epsilon_r^2/(w^2/4+iz)} \\
&\approx z + \sum_{r=1}^{\infty} \left(\frac{1}{\epsilon_r}\right) \sin(2\epsilon_r z) e^{-4\epsilon_r^2/w^2}
\end{aligned}
$$

for the values of z we are considering.

Not itself a ϑ_3 function, but $j'(z)$ **is**:

$$\begin{aligned} j'(z) &\approx 1 + 2\sum_{r=1}^{\infty} \cos(2\epsilon_r z) e^{-\epsilon_r^2 w^2/4} \\ &= 1 + 2\sum_{r=1}^{\infty} \cos(2rz/N) e^{-r^2 w^2/(4N^2)} \\ &= \vartheta_3\left(\frac{z}{N}, e^{-w^2/(4N^2)}\right) \end{aligned}$$

Behaviour of ϑ_3 not obvious, $\because \uparrow = O(1)$ for $w \ll 2N$.

However, can use alternative notation $\vartheta(z,q) = \vartheta_3(z|\tau)$, where $q = e^{i\pi\tau}$, and apply Jacobi's imaginary transformation:

$$\vartheta_3(z|\tau) = (-i\tau)^{-\frac{1}{2}} e^{-i\tau' z^2/(\pi\tau')} \vartheta_3(z\tau'|\tau'),$$

where $\tau' = -1/\tau$.

This converts $j'(z)$ to

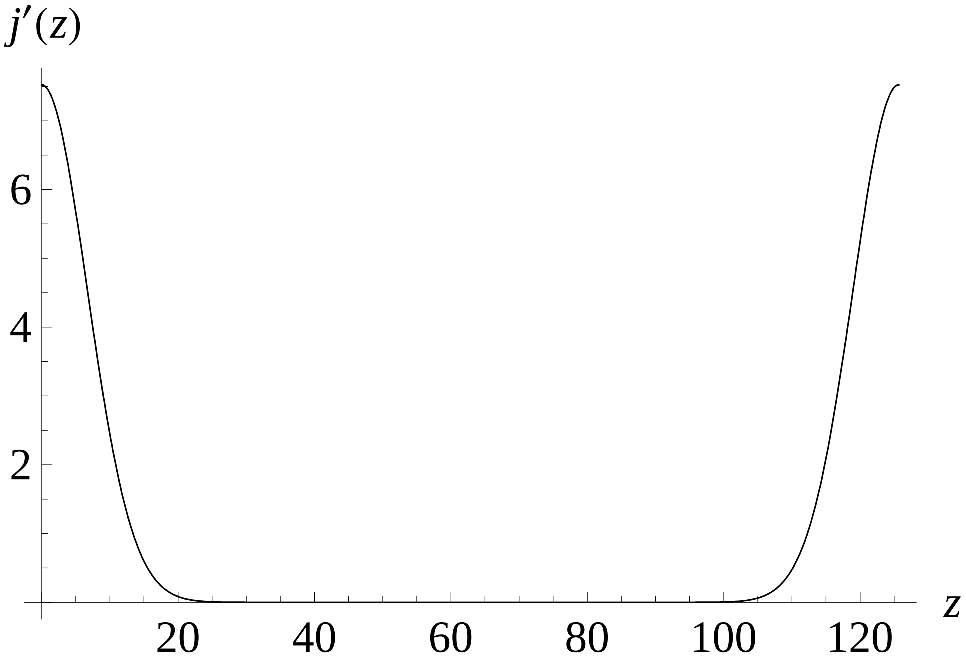
$$j'(z) = 2\sqrt{\pi}\frac{N}{w}e^{-4z^2/w^2}\vartheta_3\left(\frac{4\pi i N}{w^2}z,\ e^{-4\pi^2 N^2/w^2}\right)$$

Now second argument of ϑ_3 is small, so for reasonable z can approximate ϑ_3 by 1.

Then behaviour dominated by preceding Gaussian e^{-4z^2/w^2}, which rapidly $\downarrow$, corresponding to plateau in $j(z)$

Gaussian eventually overwhelmed by the cosh terms occurring in expansion of ϑ_3.

- Must be so, $\because$ ϑ_3 **periodic** in z .



$j'(z)$ from previous slide

6. Conclusions

- Showed how to exploit similarity transformation for unbroken PT case $\lambda < 1$

- Showed how to use method of stationary states for PT-breaking threshold $\lambda = 1$

- In this case elucidated unexpected phenomenon of saturation

- Optics very fertile ground for exploitation of PT symmetry.

- Interesting and useful properties

- Can exploit possibility of switching from broken to unbroken phase

- Interesting to consider materials where n varies with z as well[†]

[†] Lin et al.
PRL **106**, 213901 (2011)