General 2×2 $\mathcal{PT}$-Symmetric Matrices and Jordan Blocks[1]

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Quantum Physics with Non-Hermitian Operators
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[1] Work in progress with Uwe Günther and Jia-wen Deng

Outline

Definitions of $\mathcal{P}$ and $\mathcal{T}$

- Both are involution: $\mathcal{P}^2 = \mathbf{1}$ & $\mathcal{T}^2 = \mathbf{1}$.
- They commute: $[\mathcal{P}, \mathcal{T}] = 0$.
- Two choices for $\mathcal{T}$

"$\mathcal{PT}$ Symmetry"

$$\mathcal{T}A\mathcal{T} \equiv A^* \quad \Rightarrow \quad \mathcal{P} = \mathcal{P}^*$$

$$[H, \mathcal{PT}] = 0 \quad \Leftrightarrow \quad \mathcal{P}H\mathcal{P} = H^*$$

"$\mathcal{P}$ pseudo-Hermiticity"

$$\mathcal{T}A\mathcal{T} \equiv A^\dagger \quad \Rightarrow \quad \mathcal{P} = \mathcal{P}^\dagger$$

$$[H, \mathcal{PT}] = 0 \quad \Leftrightarrow \quad \mathcal{P}H\mathcal{P} = H^\dagger$$

Inner Products

- Inner products in QM [Ballentine, *Quantum Mechanics*]
  1. (ψ, ϕ) is a complex number,
  2. $(\psi, \phi) = (\phi, \psi)^*$, where $*$ denotes complex conjugate,
  3. $(\psi, c_1\phi_1 + c_2\phi_2) = c_1(\psi, \phi_1) + c_2(\psi, \phi_2)$, where c_1 and c_2 are complex numbers,
  4. $(\phi, \phi) \geq 0$, with equality holding iff $\phi = 0$.
- In general, $(\psi, \phi) \equiv \langle\psi|W|\phi\rangle$.
  1. The metric operator is a Hermitian matrix: $W = W^\dagger$
  2. All the eigenvalues of W are positive: $\lambda^W > 0$.
- A self-adjoint operator in finite dimensions

$$(\psi, H\phi) = (H\psi, \phi) \quad \Rightarrow \quad WH = H^\dagger W.$$

Definitions of $\mathcal{T}$ and $\mathcal{P}$

- Time reversal

$$\mathcal{T} \equiv \dagger \qquad \Leftrightarrow \qquad \mathcal{T} A \mathcal{T} = A^\dagger$$

- Parity

$$[\mathcal{P}, \mathcal{T}] = 0 \qquad \Rightarrow \qquad \mathcal{P} = \mathcal{P}^\dagger$$

$$\mathcal{P}(\theta, \varphi) = \mathbf{n}^r \cdot \boldsymbol{\sigma} = \begin{bmatrix} \cos\theta & \sin\theta \ \mathrm{e}^{-\mathrm{i}\varphi} \\ \sin\theta \ \mathrm{e}^{\mathrm{i}\varphi} & -\cos\theta \end{bmatrix},$$

where $\mathbf{n}^r \equiv (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$.

- Eigenvalues of $\mathcal{P}$: $\lambda^{\mathcal{P}} = \pm 1$.

The Most General $\mathcal{P}$-pseudo-Hermitian H

- Hamiltonian

$$H = e\mathbf{1} + \left(\gamma\,\mathbf{n}^r + \mathrm{i}\rho\sin\delta\,\mathbf{n}^\theta + \mathrm{i}\rho\cos\delta\,\mathbf{n}^\varphi\right)\cdot\boldsymbol{\sigma}$$
$$= \begin{bmatrix} e+\gamma\cos\theta - \mathrm{i}\rho\sin\theta\sin\delta & (\gamma\sin\theta + \mathrm{i}\rho\cos\theta\sin\delta + \rho\cos\delta)\mathrm{e}^{-\mathrm{i}\varphi} \\ (\gamma\sin\theta + \mathrm{i}\rho\cos\theta\sin\delta - \rho\cos\delta)\mathrm{e}^{\mathrm{i}\varphi} & e-\gamma\cos\theta + \mathrm{i}\rho\sin\theta\sin\delta \end{bmatrix}$$

  with

$$\begin{aligned} \mathbf{n}^\theta &\equiv (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta) \\ \mathbf{n}^\varphi &\equiv (-\sin\varphi, \cos\varphi, 0) \end{aligned}$$

- Eigenvalues

$$\lambda_\pm^H = e \pm \sqrt{\gamma^2 - \rho^2}$$

- Eigenstates

$$H|E_\pm\rangle = \lambda_\pm^H|E_\pm\rangle$$

- Hermitian limit: $\rho = 0$. All 2×2 Hermitian matrices are $\mathcal{P}(\theta,\varphi)$-pseudo-Hermitian.

The Metric Operator W

- The self-adjointness of $H \Rightarrow$ a dynamic W:

$$WH = H^{\dagger}W$$

- The metric operator

$$W = u\left[\gamma\mathbf{1} + \left(v\,\mathbf{n}^r + \rho\cos\delta\,\mathbf{n}^\theta - \rho\sin\delta\,\mathbf{n}^\varphi\right)\cdot\boldsymbol{\sigma}\right]$$

with $u\gamma > 0$ & $v^2 < \gamma^2 - \rho^2$

- Eigenvalues of W

$$\lambda^W = u\left[\gamma \pm \sqrt{\rho^2 + v^2}\right] > 0$$

- With a proper choice of u & v, $W = \mathcal{PC}$.

The Inner Product

- Definition: $(\psi, \phi)_W \equiv \langle\psi|W|\phi\rangle$
- Orthogonality: $\langle E_+|W|E_-\rangle = 0 = \langle E_-|W|E_+\rangle$
- Normalization

$$\begin{aligned}\mathcal{N}_\pm &\equiv \langle E_\pm|W|E_\pm\rangle \\ &= |n_\pm|^2 u\sqrt{\gamma^2-\rho^2}\left(\gamma \pm \sqrt{\gamma^2-\rho^2}\right)\left(\sqrt{\gamma^2-\rho^2} \pm v\right) \\ &> 0\end{aligned}$$

- $\mathcal{P}$-inner product defines a Krein space:
  - Orthogonality: $\langle E_+|\mathcal{P}|E_-\rangle = 0 = \langle E_-|\mathcal{P}|E_+\rangle$
  - But $\langle E_+|\mathcal{P}|E_+\rangle$ and $\langle E_-|\mathcal{P}|E_-\rangle$ have opposite signs.

Jordan Blocks

- Condition: $\gamma^2 = \rho^2 \neq 0$
- Assume $\gamma = \rho$
- Only one eigenstate: $H|\Phi_0\rangle = e|\Phi_0\rangle$

$$|\Phi_0\rangle = n_0 \begin{bmatrix} \cos\frac{\theta}{2} - \mathrm{e}^{-\mathrm{i}(\delta+\varphi)}\sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} - \mathrm{e}^{-\mathrm{i}(\delta+\varphi)}\cos\frac{\theta}{2} \end{bmatrix}$$

- Jordan chain:

$$(H - e\mathbf{1})|\Phi_1\rangle = |\Phi_0\rangle$$

$$\Downarrow$$

$$|\Phi_1\rangle = n_0 \frac{\mathrm{e}^{-\mathrm{i}(\delta+\varphi)}}{\gamma} \begin{bmatrix} \sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{bmatrix} + \alpha|\Phi_0\rangle$$

with arbitrary α.

What goes wrong when H approaches a Jordan block?

- Close to a Jordan block: $\rho^2 \equiv \gamma^2(1-\epsilon) \neq 0$, where $0 < \epsilon \ll 1$
- For simplicity, fix u and set $v = 0$.
- What happens to W?
  - The larger eigenvalue of W: $\lambda_>^W \sim 2u\gamma$
  - The smaller eigenvalue of W: $\lambda_<^W \sim \frac{1}{2}u\gamma\epsilon$
  - W stops being positive definite.
- How about eigenstates?
  - Normalization: $\mathcal{N}_\pm \sim |n_\pm|^2 u\gamma^3\epsilon$
  - Eigenstates cannot be normalized.

Definitions of $\mathcal{T}$ and $\mathcal{P}$

- Time reversal

$$\mathcal{T} \equiv * \qquad \Leftrightarrow \qquad \mathcal{T} A \mathcal{T} = A^*$$

- Parity is real: $[\mathcal{P}, \mathcal{T}] = 0 \qquad \Rightarrow \qquad \mathcal{P} = \mathcal{P}^*$
- Parity #1

$$\mathcal{P}_1 = \begin{bmatrix} \cos\theta & \sin\theta \ \mathrm{e}^{-\varphi} \\ \sin\theta \ \mathrm{e}^{\varphi} & -\cos\theta \end{bmatrix}$$

- Parity #2

$$\mathcal{P}_2 = \begin{bmatrix} \cosh\theta & \sinh\theta \ \mathrm{e}^{-\varphi} \\ -\sinh\theta \ \mathrm{e}^{\varphi} & -\cosh\theta \end{bmatrix}$$

- Eigenvalues of $\mathcal{P}$: $\lambda^{\mathcal{P}} = \pm 1$.

The Most General $\mathcal{PT}$-Symmetric H

- Hamiltonian #1

$$H_1 = \begin{bmatrix} e + \gamma\cos\theta\cos\delta - \mathrm{i}\rho\sin\theta & (\gamma\sin\theta\cos\delta - \mathrm{i}\gamma\sin\delta + \mathrm{i}\rho\cos\theta)\mathrm{e}^{-\varphi} \\ (\gamma\sin\theta\cos\delta + \mathrm{i}\gamma\sin\delta + \mathrm{i}\rho\cos\theta)\mathrm{e}^{\varphi} & e - \gamma\cos\theta\cos\delta + \mathrm{i}\rho\sin\theta \end{bmatrix}$$

- Hamiltonian #2

$$H_2 = \begin{bmatrix} e + \gamma\cos(\delta + \mathrm{i}\theta) & -\mathrm{i}[\gamma\sin(\delta + \mathrm{i}\theta) - \rho]\mathrm{e}^{-\varphi} \\ \mathrm{i}[\gamma\sin(\delta + \mathrm{i}\theta) + \rho]\mathrm{e}^{\varphi} & e - \gamma\cos(\delta + \mathrm{i}\theta) \end{bmatrix}$$

- Eigenvalues

$$\lambda_\pm^H = e \pm \sqrt{\gamma^2 - \rho^2}$$

- Eigenstates of H: $H|E_\pm\rangle = \lambda_\pm^H |E_\pm\rangle$
- When $\mathcal{PT}$ symmetry is not broken ($\rho^2 \leq \gamma^2$), they are also the eigenstates of $\mathcal{PT}$: $\mathcal{PT}|E_\pm\rangle \equiv \mathcal{P}|E_\pm\rangle^* = \lambda_\pm^{\mathcal{PT}} |E_\pm\rangle$.

Hermitian Limit

- Hamiltonian #1: $\rho = \phi = 0$
  - $H_1 \to \begin{bmatrix} e + \gamma\cos\theta\cos\delta & \gamma\sin\theta\cos\delta - \mathrm{i}\gamma\sin\delta \\ \gamma\sin\theta\cos\delta + \mathrm{i}\gamma\sin\delta & e - \gamma\cos\theta\cos\delta \end{bmatrix}$
  - All 2×2 Hermitian matrices are $\mathcal{P}_1\mathcal{T}$-symmetric.
- Hamiltonian #2: $\rho = \phi = \theta = 0$
  - $H_2 \to \begin{bmatrix} e + \gamma\cos\delta & -\mathrm{i}\gamma\sin\delta \\ \mathrm{i}\gamma\sin\delta & e - \gamma\cos\delta \end{bmatrix}$
  - Only some Hermitian matrices are $\mathcal{P}_2\mathcal{T}$-symmetric.
- Hermitian H_2 is just a special case of Hermitian H_1 with $\theta = 0$.
- H_1 and H_2 coincide when $\phi = \theta = 0$.

The Metric W

- The self-adjointness of H $\Rightarrow$ a dynamic W:

$$WH = H^{\dagger}W$$

- Metric operator #1

$$W_1 = u\begin{bmatrix} [\gamma + \cos\theta(\rho\sin\delta + v\cos\delta)]\,\mathrm{e}^{\varphi} & \sin\theta(\rho\sin\delta + v\cos\delta) + \mathrm{i}(\rho\cos\delta - v\sin\delta) \\ \sin\theta(\rho\sin\delta + v\cos\delta) - \mathrm{i}(\rho\cos\delta - v\sin\delta) & [\gamma - \cos\theta(\rho\sin\delta + v\cos\delta)]\,\mathrm{e}^{-\varphi} \end{bmatrix}$$

- Metric operator #2

$$W_2 = u\begin{bmatrix} [\gamma\cosh\theta + (\rho\sin\delta + v\cos\delta)]\,\mathrm{e}^{\varphi} & \gamma\sinh\theta + \mathrm{i}(\rho\cos\delta - v\sin\delta) \\ \gamma\sinh\theta - \mathrm{i}(\rho\cos\delta - v\sin\delta) & [\gamma\cosh\theta - (\rho\sin\delta + v\cos\delta)]\,\mathrm{e}^{-\varphi} \end{bmatrix}$$

- Both with $u\gamma > 0$ & $v^2 < \gamma^2 - \rho^2$
- Eigenvalues of W

$$\lambda^W = u\left[\gamma \pm \sqrt{\rho^2 + v^2}\right] > 0$$

The Inner Product

- Definition: $(\psi, \phi)_W \equiv \langle\psi|W|\phi\rangle$
- Orthogonality

$$\langle E_+|W|E_-\rangle = 0 = \langle E_-|W|E_+\rangle$$

- Normalization

$$\begin{aligned}\mathcal{N}_\pm &\equiv \langle E_\pm|W|E_\pm\rangle \\ &= |n_\pm|^2 u\gamma\sqrt{\gamma^2-\rho^2}\left(\sqrt{\gamma^2-\rho^2}\pm v\right) \\ &> 0\end{aligned}$$

Jordan Blocks

- Condition: $\gamma^2 = \rho^2 \neq 0$
- Assume $\gamma = \rho$
- One eigenstate: $H_1|\Phi_0\rangle = e|\Phi_0\rangle$
  - $|\Phi_0\rangle = n_0 \begin{bmatrix} \cos\frac{\theta}{2}(1-\sin\delta) + \mathrm{i}e^{-\varphi}\sin\frac{\theta}{2}\cos\delta \\ -\sin\frac{\theta}{2}(1-\sin\delta) + \mathrm{i}e^{-\varphi}\cos\frac{\theta}{2}\cos\delta \end{bmatrix}$
  - It is also an eigenstate of $\mathcal{PT}$: $\mathcal{PT}|\Phi_0\rangle = \frac{n_0^*}{n_0}|\Phi_0\rangle$
- The Jordan chain: $(H_1 - e\mathbf{1})|\Phi_1\rangle = |\Phi_0\rangle$
  - $|\Phi_1\rangle = n_0\frac{1-\sin\delta}{\gamma\cos\delta}\begin{bmatrix} \cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2} \end{bmatrix} + \alpha|\Phi_0\rangle$ with arbitrary α.
  - When α is real, $|\Phi_1\rangle$ is also an eigenstate of $\mathcal{PT}$ with same eigenvalue, $\mathcal{PT}|\Phi_1\rangle = \frac{n_0^*}{n_0}|\Phi_1\rangle$
- Similar results for Case #2.

What goes wrong when H approaches a Jordan block?

- Exactly the same thing happens.
- Close to a Jordan block: $\rho^2 \equiv \gamma^2(1-\epsilon) \neq 0$, where $0 < \epsilon \ll 1$
- For simplicity, fix u and set $v = 0$.
- What happens to W?
  - The larger eigenvalue of W: $\lambda_>^W \sim 2u\gamma$
  - The smaller eigenvalue of W: $\lambda_<^W \sim \frac{1}{2}u\gamma\epsilon$
  - W stops being positive definite.
- How about eigenstates?
  - Normalization: $\mathcal{N}_\pm \sim |n_\pm|^2 u\gamma^3\epsilon$
  - Eigenstates cannot be normalized.

Concluding Remarks

- All 2×2 Hermitian matrices are both $\mathcal{P}$-pseudo-Hermitian and $\mathcal{PT}$-symmetric with respect to some $\mathcal{P}$.
- In $\mathcal{P}$ pseudo-Hermiticity, $\mathcal{P}$ can be used to define a Krein space.
- When $\mathcal{PT}$ symmetry is not broken, eigenstates of $\mathcal{PT}$-symmetric H are also eigenstates of $\mathcal{PT}$.
- Both $\mathcal{P}$-pseudo-Hermitian and $\mathcal{PT}$-symmetric matrices may form Jordan block.
- When H forms a Jordan block, W becomes ill-defined and the eigenstates cannot be normalized.