Perfect transmission scattering as a $\mathcal{PT}$-symmetric spectral problem

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Joint work with:
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D. Krejčiřík (Doppler Institute Prague & BCAM, Bilbao)
J. Železný (Doppler Institute Prague)

Operator

Hilbert space and operator

- $\mathcal{H} = L^2(-a, a)$
- $\boxed{H = -\Delta + V}$ $\boxed{\psi'(\pm a) + c_\pm \psi(\pm a) = 0}$
- $V \in \mathscr{B}(\mathcal{H})$ (not only potential), $c_\pm \in \mathbb{C}$

Remarks

- $H^* = -\Delta + V^*$
  $\psi'(\pm a) + \overline{c_\pm}\psi(\pm a) = 0$
- H is non-Hermitian
- H is $\mathcal{PT}$-symmetric iff $c_\pm = i\alpha \pm \beta$ and $[V, PT] = 0$
- $H = \mathcal{T} H^* \mathcal{T}$ iff $V = \mathcal{T} V^* \mathcal{T}$

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Outline

Two parts

1. QM reflectionless scattering
2. Non-locality

QM Reflectionless Scattering

- 2011 H. Hernandez-Coronado, D. Krejčiřík, P. Siegl
  Perfect transmission scattering as a $\mathcal{PT}$-symmetric spectral problem
  Physics Letters A 375.
- ! description of reflectionless scattering via spectra of (not only) $\mathcal{PT}$-symmetric operators
- ! complex eigenvalues have very natural physical interpretation

Non-locality

- 2011 D. Krejčiřík, P. Siegl, and J. Železný
  On the similarity of Sturm-Liouville operators with non-Hermitian boundary conditions to self-adjoint and normal operators
- ! metric Θ, $\mathcal{C}$-operator, similarity transformation Ω, similar operator h
- ! existence, structure, example of H with Θ, $\mathcal{C}$, Ω, h in closed form

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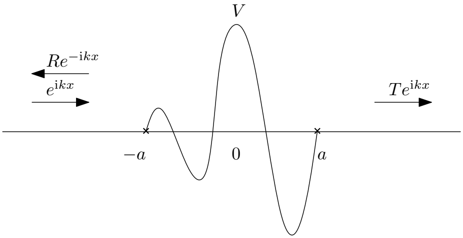
1. QM Reflectionless Scattering

Model

Scattering in compactly supported potential V

- 1D particle on line, $\mathcal{H} = L^2(\mathbb{R})$
- real potential V with support in $(-a, a)$



- asymptotic solutions of Schrödinger equation

$$-\psi'' + V\psi = k^2\psi$$

$$\psi_{\mathrm{as}}(x) = \begin{cases} e^{ikx} + Re^{-ikx}, & x < -a \\ Te^{ikx}, & x > a \end{cases}$$

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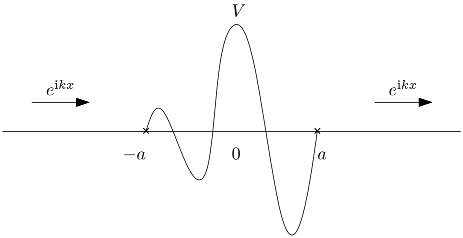

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Model

How to solve reflectionless scattering?

- Schrödinger equation

$$-\psi'' + V\psi = k^2\psi$$

- asymptotic solution

$$\psi_{\mathrm{as}}(x) = \begin{cases} e^{\mathrm{i}kx}, & x < -a \\ e^{\mathrm{i}kx}, & x > a \end{cases}$$

- matching to the solution ψ in $(-a, a)$

$$\psi(-a) = \psi_{\mathrm{as}}(-a), \quad \psi'(-a) = \psi'_{\mathrm{as}}(-a) = \mathrm{i}k\psi(-a)$$
$$\psi(a) = \psi_{\mathrm{as}}(a), \quad \psi'(a) = \psi'_{\mathrm{as}}(a) = \mathrm{i}k\psi(a)$$

- reformulation in $L^2(-a, a)$

$$\begin{cases} -\psi''(x) + V(x)\psi(x) & = k^2\psi(x), \quad x \in (-a, a) \\ \psi'(\pm a) - \mathrm{i}\,k\,\psi(\pm a) & = 0 \end{cases}$$

- nonlinear problem with energy dependent transparent BC

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From non-linear to non-Hermitian spectral problem

- non-linear problem

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- new parameter $\alpha \in \mathbb{R}$

$$\begin{cases} -\psi''(x) + V(x)\psi(x) &= \mu(\alpha)\psi(x), \quad x \in (-a,a) \\ \psi'(\pm a) + \mathrm{i}\, \alpha\, \psi(\pm a) &= 0 \end{cases}$$

+ additional condition

$$\mu(\alpha) = \alpha^2$$

⇒ parametric spectral problem + "dispersion relation"

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$$\begin{cases} H_\alpha\psi & = \mu(\alpha)\psi \\ \mu(\alpha) & = \alpha^2 \end{cases}$$

$\mathcal{PT}$-symmetry

Appearance of $\mathcal{PT}$-symmetry

- Operator H_α
  - $H_\alpha = -\Delta + V$

  BC: $\psi'(\pm a) + \mathrm{i}\,\alpha\,\psi(\pm a) = 0$
- $H_\alpha^* = H_{-\alpha}$

! H_α is $\mathcal{PT}$-symmetric if $V(x) = V(-x)$

Strategy

- find the eigenvalues of associated operator H_α as function of α
- select eigenvalues lying on parabola α^2
- solution: perfect transmission energies

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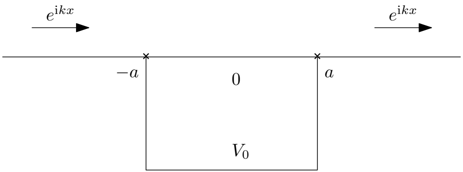
Examples

Square well

- $V(x) = V_0$



- associated spectral problem

$$\begin{cases} -\psi''(x) + V_0\psi(x) & = \mu(\alpha)\psi(x) \\ \psi'(\pm a) + \mathrm{i}\,\alpha\,\psi(\pm a) & = 0 \end{cases}$$

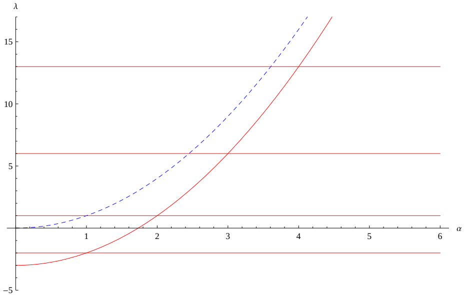
Examples

Square well

- eigenvalues of $H(\alpha)$:

$$\mu_0(\alpha) = \alpha^2 + V_0,$$
$$\mu_n(\alpha) = (n\pi/2a)^2 + V_0$$

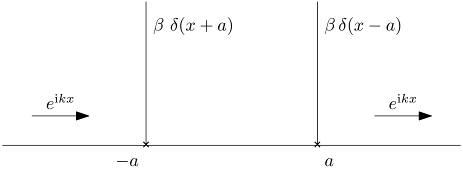
- PTEs: $k_n^2 = (n\pi/2a)^2 - V_0$
- if $V_0 = 0$, then PTEs: $k^2 \in \mathbb{R}^+$



Examples

Two δ potentials

- $V(x) = \beta(\delta(x+a) + \delta(x-a))$



- associated spectral problem

$$\begin{cases} -\psi''(x) & = \mu(\alpha)\psi(x), \\ \psi'(\pm a) + (\mathrm{i}\,\alpha \pm \beta)\,\psi(\pm a) & = 0 \end{cases}$$

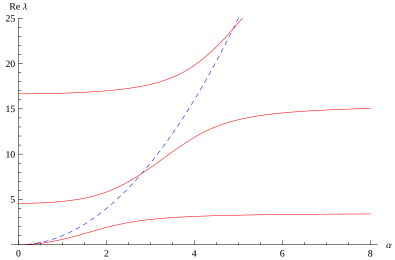
Two δ potentials

$\beta > 0$

- Eigenvalue equation

$$(k^2 - \alpha^2 - \beta^2)\sin 2ka - 2\beta k \cos 2ka = 0$$

- all eigenvalues are real [KrSi10]



[KrSi10] 2010 Krejčiřík, Siegl, *Journal of Physics A: Mathematical and General* 43

Two δ potentials

$\beta < 0$

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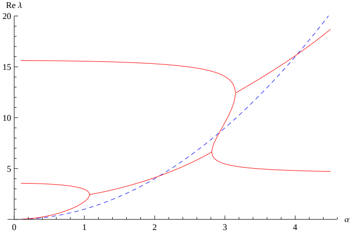
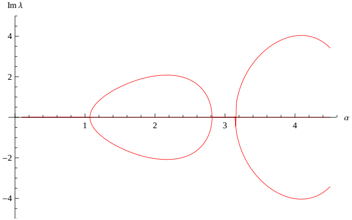
- either one or no pair of complex conjugated eigenvalues [KrSi10]





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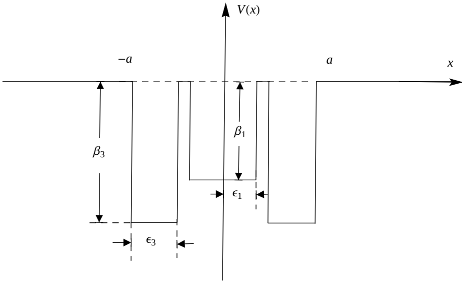
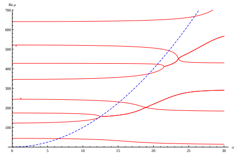
Interpretation of complex EVs

Three wells potential



Spectral picture

$a = \pi/4,\ \varepsilon_1 = 0.2,\ \varepsilon_3 = 0.5,\ \beta_1 = -90,\ \beta_3 = -100$



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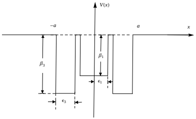

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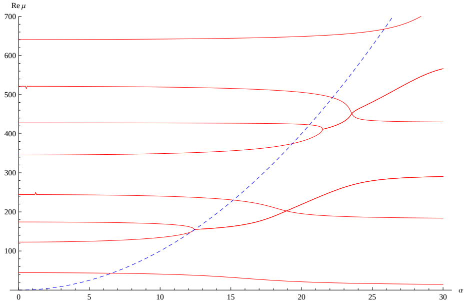
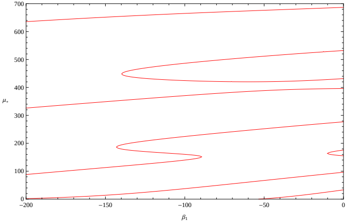


Interpretation of complex EVs

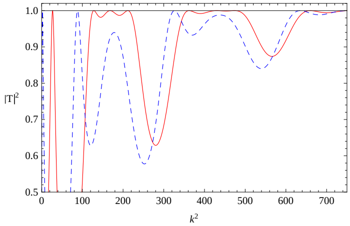
PTEs as a function of the potential

$a = \pi/4$, $\varepsilon_1 = 0.2$, $\varepsilon_3 = 0.5$, $\beta_3 = -100$, and $\beta_2 = 0$

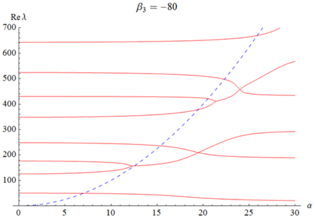


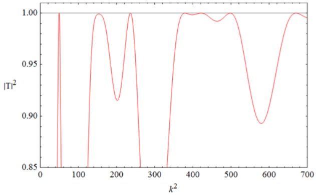
Transmission coefficient

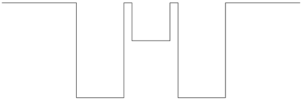
$\beta_1 = -120$ (cont. red), and $\beta_1 = -200$ (dashed blue)

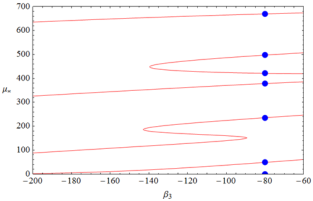


Animation


$\beta_3 = -80$
Re λ
700
600
500
400
300
200
100
0
5
10
15
20
25
30
α


1.00
0.95
0.90
0.85
$|T|^2$
0
100
200
300
400
500
600
700
k^2




700
600
500
400
300
200
100
0
μ_*
−200
−180
−160
−140
−120
−100
−80
−60
β_3

Inverse problem

- what can we say about the spectrum of

$$\begin{cases} -\psi''(x) + V(x)\psi(x) = \mu(\alpha)\psi(x), & x \in (-a,a) \\ \psi'(\pm a) - \mathrm{i}\,\alpha\,\psi(\pm a) = 0 \end{cases} \quad ?$$

Add constant potential V_0

-

$$\begin{cases} -\psi''(x) + (V(x) + V_0)\psi(x) = \mu(V_0,\alpha)\psi(x), & x \in (-a,a) \\ \psi'(\pm a) - \mathrm{i}\,\alpha\,\psi(\pm a) = 0 \end{cases}$$

$$\mu(V_0,\alpha) = \alpha^2$$

- V_0 shifts the EVs up and down

× parabola determining PTEs is not shifted

- appearance of complex conjugated pair

⇔ loss of two close PTEs

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Notations

Operator

$$H = -\Delta + V$$
$$\psi'(\pm a) + c_{\pm}\psi(\pm a) = 0$$

Eigenvalues, eigenfunctions

$$H\psi_n = E_n\psi_n$$
$$H^*\phi_n = \overline{E_n}\phi_n$$

Metric operator Θ, similarity transformation Ω

$$\Theta := \sum_{n=0}^{\infty} c_n\phi_n\langle\phi_n,\cdot\rangle$$
$$\Theta = \Omega^*\Omega$$
$$h := \Omega H \Omega^{-1}$$

Similarity to Hermitian and normal operator [Si11]

$$E_n \in \mathbb{R} \quad \Leftrightarrow \quad \Theta H = H^*\Theta \quad \Leftrightarrow \quad h = h^*$$
$$E_n \notin \mathbb{R} \quad \Leftrightarrow \quad \Theta H \Theta^{-1} H^* = H^*\Theta H \Theta^{-1} \quad \Leftrightarrow \quad hh^* = h^*h$$

[Si11] 2011 Siegl, *IJTP* 50

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Similarity transformation Ω

Metric operator Θ

$$\Theta = \sum_{n=0}^{\infty} \phi_n \langle \phi_n, \cdot \rangle$$

Similarity transformation Ω

- Factorization

$$\Theta = \Omega^* \Omega$$

- take arbitrary orthonormal basis $\{e_n\}$ in $\mathcal{H}$
- define

$$\Omega := \sum_{n=0}^{\infty} e_n \langle \phi_n, \cdot \rangle$$

- properties

$$\Omega : \psi_n \mapsto e_n$$
$$\Omega^* : e_n \mapsto \phi_n$$
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Θ, Ω, and h

Existence of Θ for H

- reformulation of known results [Mi62], [DSIII]
- bounded, positive, and invertible metric operator Θ exists

$\Leftrightarrow$ all eigenvalues of H are simple

- multiple EVs $\Leftrightarrow$ Jordan blocks $\Leftrightarrow$ Θ is not invertible
- existence of Ω follows
- $h = \Omega H \Omega^{-1}$ is Hermitian iff $\sigma(H) \subset \mathbb{R}$
- complex eigenvalues $\Rightarrow$ h is normal

[DSIII] 1971 Dunford, Schwartz, *Linear Operators, Part 3, Spectral Operators*

[Mi62] 1962 Mikhajlov, *Dokl. Akad. Nauk SSSR* 114

Structure of Θ and Ω

Theorem

Let all eigenvalues of H be simple and let $\{e_n\}$ be an orthonormal basis of $\mathcal{H}$. Then

- $\Theta = I + K$,
- $\Omega = U + L$,

where K, L are integral (H-S) operators and U is a unitary operator. Moreover, if $e_n := \chi_n^N$ (eigenfunctions of $-\Delta_N$), then $U = I$.

Remarks

- proof: asymptotics of EVs and EFs + analytic perturbation theory
- kernels of K, L are in $L^2((-a,a)\times(-a,a))$
- $\Omega^{-1} = U^{-1} + M$, M is an integral operator
- similar h is typically non-local
- "preferred" basis χ_n^N
- $\mathcal{PT}$-symmetry is not needed (but provides "nice" examples)
- valid for strongly regular connected BC as well (expected)
- K is not always an integral operator (compact): $\Theta = I - \mathrm{i}\sin\phi\mathcal{P}\,\mathrm{sgn}(x)\cdot$ for "$\mathcal{PT}$-quasi-periodic BC" [Si08]

[Si08] 2008 Siegl, *Journal of Physics A: Mathematical and Theoretical* 41

Structure of Θ and Ω

Theorem

Let all eigenvalues of H be simple and let $\{e_n\}$ be an orthonormal basis of $\mathcal{H}$. Then

- $\Theta = I + K,$
- $\Omega = U + L,$

where K, L are integral (H-S) operators and U is a unitary operator. Moreover, if $e_n := \chi_n^N$ (eigenfunctions of $-\Delta_N$), then $U = I$.

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[Si08] 2008 Siegl, *Journal of Physics A: Mathematical and Theoretical* 41

$\mathcal{PT}$-symmetric example I

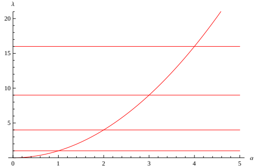
Operator [KrBiZn06]

$$H_\alpha = -\Delta$$
$$\psi'(\pm a) + \mathrm{i}\alpha\psi(\pm a) = 0$$

Eigenvalues and eigenfunctions

$\sigma(H) = \{\alpha^2\} \cup \{k_n^2\}_{n=1}^{\infty}$, $k_n = n\pi/(2a)$

$$\psi_0(x) = A_0 e^{-\mathrm{i}\alpha(x+a)}, \qquad \psi_n(x) = A_n \left(\chi_n^N(x) - \mathrm{i}\frac{\alpha}{k_n}\chi_n^D(x) \right),$$
$$\phi_0(x) = \frac{1}{\sqrt{2a}}\, e^{\mathrm{i}\alpha(x+a)}, \qquad \phi_n(x) = \chi_n^N(x) + \mathrm{i}\frac{\alpha}{k_n}\chi_n^D(x),$$



[KrBiZn06] 2006 Krejčiřík, Bíla, Znojil, *Journal of Physics A: Mathematical and General* 39

Θ and Ω operators

Formulae

$$\Omega = \sum_{n=0}^{\infty} \chi_n^N \langle \phi_n, \cdot \rangle, \quad \Theta = \sum_{n=0}^{\infty} \phi_n \langle \phi_n, \cdot \rangle$$

$$\phi_n = \chi_n^N + \mathrm{i}\frac{\alpha}{k_n}\chi_n^D, \quad \phi_0(x) = \frac{1}{\sqrt{2a}}\, e^{\mathrm{i}\alpha(x+a)}$$

Construction of Ω

$$\Omega = \sum_{n=0}^{\infty} \chi_n^N \langle \phi_n, \cdot \rangle = \sum_{n=1}^{\infty} \chi_n^N \langle \chi_n^N, \cdot \rangle - \mathrm{i}\frac{\alpha}{k_n}\sum_{n=1}^{\infty} \chi_n^N \langle \chi_n^D, \cdot \rangle$$
$$+ \chi_0^N \langle \phi_0^N, \cdot \rangle + \chi_0^N \langle \chi_0^N, \cdot \rangle - \chi_0^N \langle \chi_0^N, \cdot \rangle$$

$$= \sum_{n=0}^{\infty} \chi_n^N \langle \chi_n^N, \cdot \rangle + \chi_0^N \langle \phi_0^N - \chi_0^N, \cdot \rangle + \alpha p \sum_{n=0}^{\infty} \frac{1}{k_n^2} \chi_n^D \langle \chi_n^D, \cdot \rangle$$

$$= I + \chi_0^N \langle \phi_0^N - \chi_0^N, \cdot \rangle + \alpha p (-\Delta_D)^{-1}$$

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Θ and Ω operators

Proposition (Θ)

Any Θ for H_α has the form

$$\Theta = J^N + c_0\theta_1 + J^N\theta_2 + J^D\theta_3$$

$c_0 \in \mathbb{R}^+$, θ_i are integral operators with kernels:

$$\theta_1(x,y) := \frac{\mathrm{i}}{a} e^{\frac{\mathrm{i}\alpha}{2}(y-x)} \sin\left(\frac{\alpha}{2}(y-x)\right),$$

$$\theta_2(x,y) := \frac{\mathrm{i}\alpha}{2a}\left(y - a\,\mathrm{sgn}\,(y-x)\right),$$

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and

$$J^D := \sum_{n=1}^{\infty} c_n \chi_n^D \langle \chi_n^D, \cdot\rangle, \quad J^N := \sum_{n=0}^{\infty} c_n \chi_n^N \langle \chi_n^N, \cdot\rangle$$

Remarks

- $J^{D,N} = I$ if $c_n = 1$
- $J^{D,N}$ are metric operators for $-\Delta_{D,N}$
- different explicit Θ's, construction of a dense set [Ze11]

[Ze11] 2011 J. Železný, Master Thesis

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Θ and Ω operators

Closed form of Θ, Ω, Ω^{-1}

- $\Theta = I + K, \quad \Omega = I + L, \quad \Omega^{-1} = I + M$
- K, L, M integral operators with kernels

$$\mathcal{K}(x,y) = \alpha e^{-\mathrm{i}\alpha(y-x)} \left(\tan(\alpha a) - \mathrm{i}\,\mathrm{sgn}\,(y-x)\right)$$

$$\mathcal{L}(x,y) = \frac{\mathrm{i}\alpha}{2a}\left(y - a\,\mathrm{sgn}\,(y-x)\right) + \frac{1}{2a}\left(e^{\mathrm{i}\alpha(y+a)} - 1\right)$$

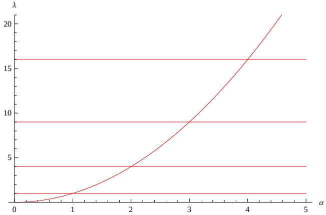
$$\mathcal{M}(x,y) = \frac{\alpha e^{\mathrm{i}\alpha(a-x)}}{\sin(2\alpha a)} - \frac{\alpha}{2} e^{-\mathrm{i}\alpha(x-y)} \left(\cot(2\alpha a) - \mathrm{i}\,\mathrm{sgn}\,(y-x)\right) - \frac{\alpha e^{-\mathrm{i}\alpha(x+y)}}{2\sin(2\alpha a)}$$

- K for special choice of $c_n \neq 1$ corresponding to $\mathcal{C}$-operator, $\mathcal{C} = \mathcal{P}\Theta$
- kernel for $c_n = 1$:

$$\tilde{\mathcal{K}}(x,y) = \frac{\mathrm{i}}{a} e^{\mathrm{i}\frac{\alpha}{2}(y-x)} \sin\left(\frac{\alpha}{2}(y-x)\right) + \frac{\alpha^2}{2a}\left(a^2 - xy\right) + \frac{\mathrm{i}\alpha}{2a}(y-x) - \frac{\mathrm{i}\alpha}{2}\left(2 - \mathrm{i}\alpha(y-x)\right)\mathrm{sgn}(y-x).$$

Similar Hermitian operator

- $h := \Omega H \Omega^{-1}$
- $h = -\Delta_N + \alpha^2 \langle \chi_0^N, \cdot \rangle \chi_0^N$
- rank one perturbation of $-\Delta_N$



Remarks

- multiple EVs $\Leftrightarrow$ Θ, Ω break down (not invertible)
- h is Hermitian (without Jordan blocks) even with multiple EVs
- all other Hermitian $\tilde{h}$ similar to H are unitarily equivalent to h

$\mathcal{PT}$-symmetric example II

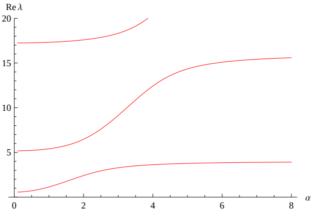
Operator [KrBiZn06]

$$H_{\alpha,\beta} = -\Delta$$
$$\psi'(\pm a) + (\mathrm{i}\alpha \pm \beta)\psi(\pm a) = 0$$

Eigenvalues

$$(k^2 - \alpha^2 - \beta^2)\sin(2ak) - 2\beta k\cos(2ak) = 0.$$

$\beta > 0$



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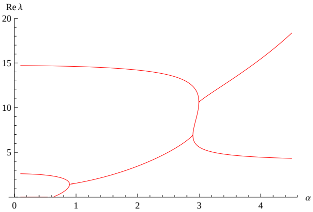

$\mathcal{PT}$-symmetric example II

Metric Operator

$$\Theta = I + K$$

with

$$\mathcal{K}(x,y) = e^{[\mathrm{i}\alpha - \beta \operatorname{sgn}(x-y)](x-y)}(c + \mathrm{i}\alpha \operatorname{sgn}(x-y)), \quad c \in \mathbb{R},$$

Remarks

- realization of $\Theta = I + K \to$ conditions on K from $\Theta H = H^*\Theta$
- Θ is positive for β small
- Ω not known

Conclusions

Summary

1. Reflectionless scattering
   - effective description via associated spectral problem
   - complex eigenvalues have natural physical interpretation
   - not necessarily only $\mathcal{PT}$-symmetric problem
2. Non-locality
   - general structure of Θ, Ω
   - factorization of $\Theta = \Omega^*\Omega$
   - explicitly solvable examples

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