Classical mechanical systems with complex potentials

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Outline of Talk :

1. Motivation to study such systems
2. Brief Introduction to the Technique followed
3. Extension to complex classical systems
4. Obtain expressions for classical trajectories, classical momenta and phase space trajectories, for a couple of explicit examples
5. Plots of trajectories & momenta
6. Discussions

Work based on :

A. Sinha, D. Dutta and P. Roy,

Phys. Lett. A, vol 375, 2011, pg 452 - 457

Q. Why study complex classical mechanics ?

As an effort to understand the classical limit of complex quantum theories.

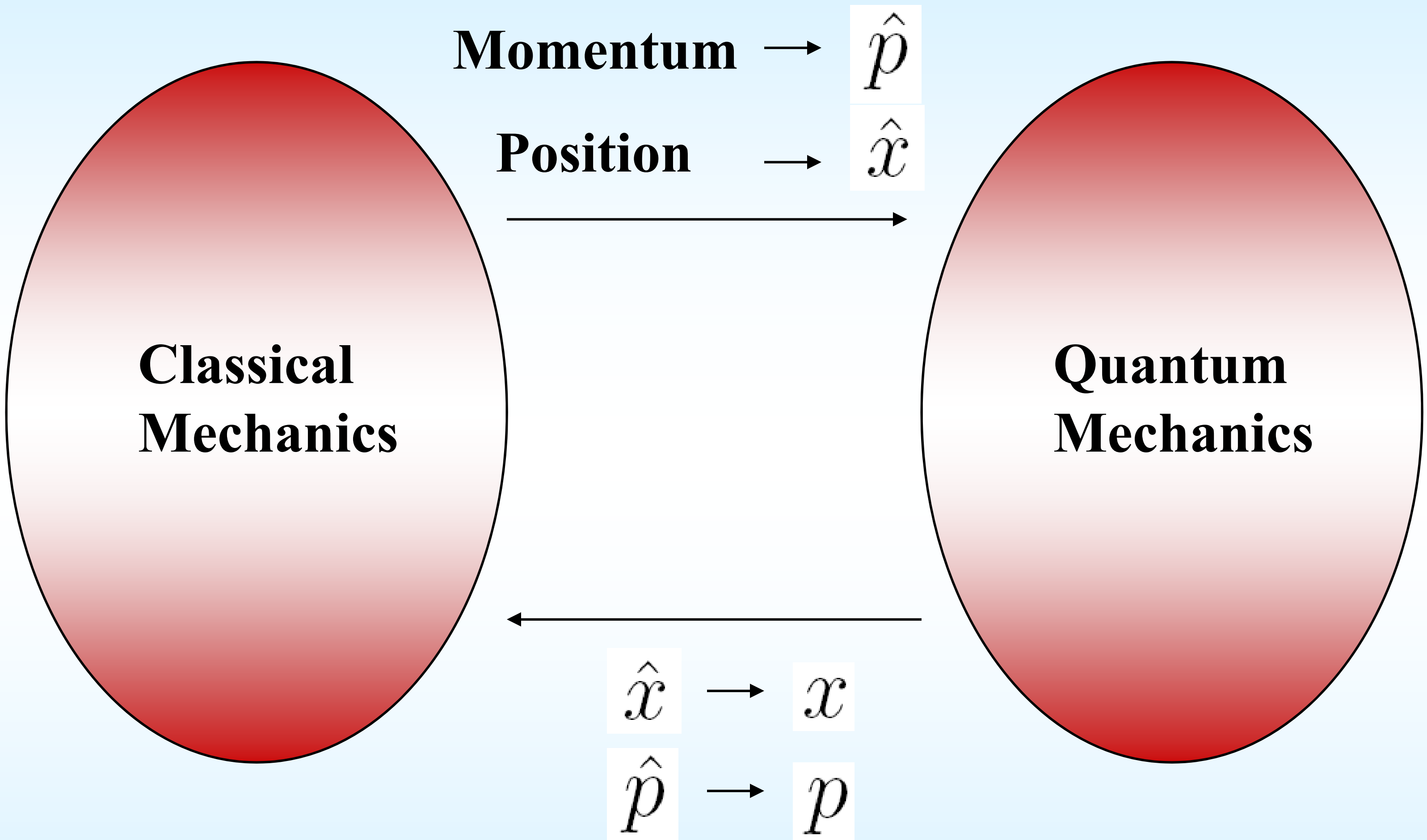
In the study of complex classical systems, the complex as well as the real solutions to Hamilton's differential equations of motion are considered.
In this generalization of conventional classical mechanics, classical particles are not constrained to move along the real axis and may travel through the complex plane.

Motivation behind extension of classical mechanics into the complex domain

To enhance one's understanding of the subtle mathematical phenomena that real physical systems can exhibit. E.g.

1. Some of the complicated properties of chaotic systems become more transparent when extended into the complex domain.
2. Studies of exceptional points of complex systems have revealed interesting and potentially observable effects .
3. The prospect of understanding the nature of tunneling.

Classical – Quantum Correspondence :



Any insight into the classical motion of complex (nH) systems ?

PT -symmetric classical mechanics —

strange dynamics of a classical particle subject to complex forces

Describes the properties of the corresponding classical theory that underlies the quantum mechanical theory described by a non Hermitian *PT* –symmetric Hamiltonian

Gives the motion of a particle that feels complex forces and responds by moving about in the complex plane

Any connection between

- **the reality of the spectrum and**
- **the regularity of the classical trajectories ?**

Observations from previous works (mainly numerical) :

- ❖ **Closed periodic orbits for unbroken *PT* symmetry**
- ❖ **Open orbits for broken *PT* symmetry (except special cases)**
- ❖ **No trajectory may cross**
- ❖ **Time period same for each orbit**
- ❖ **Orbits symmetric wrt *PT* (reflections about imaginary axis)**

Classical Hamiltonian

$$H(x,p) = p^2 + V(x)$$

Poisson bracket $\{x,p\} = 1$

Eqns of motion follow from Hamilton's eqns

$$\dot{x} = \frac{\partial H}{\partial p} = 2p$$

$$\dot{p} = -\frac{\partial H}{\partial x} = -V'(x)$$

Motion of a classical particle that feels complex forces, and moves in the complex plane, is given by

$$\ddot{x} = 2\dot{p} = -2V'(x)$$

- **velocity**

$$v = \frac{dx}{dt} = \pm 2\sqrt{E - V(x)}$$

- **Classical turning points**

$$E - V(x) = 0$$

Conventional classical mechanics :

the only possible initial positions for the particle are on the real-x axis between the turning points because the velocity is real ; all other points on the real axis belong to the so-called classically forbidden region.

However, because we are analytically continuing classical mechanics into the complex plane, we can choose any point in the complex plane as an initial position

• Factorization Technique

$$H = A^+ A^- + \gamma(H)$$

S. Kuru & J. Negro Annals of Physics **323** (2008) p 413–431

In usual factorizations in QM, $\gamma(H)$ is factorization energy. In this approach, $\gamma(H)$ may depend on *H*

$$A^{\pm} = \mp i f(x) p + \sqrt{H} g(x) + \varphi(x) + \phi(H)$$

In this approach, time-dependent integrals of motion are used to study stationary systems

The Hamiltonians that allow for this treatment are classical analogues of some quantum systems. The algebraic structure of these quantum and classical systems are similar, but with some differences.

$$A^{\pm} \text{ and } H$$

are assumed to define a deformed algebra

$$\begin{aligned} \left\{A^{\pm}, H\right\} &= \pm i\alpha(H)A^{\pm} \\ \left\{A^{+}, A^{-}\right\} &= -i\beta(H)A^{\pm} \end{aligned}$$

The auxiliary functions $\alpha(H)$, $\beta(H)$ and $\varphi(H)$ are expressed in terms of the powers of $\sqrt{H}$

In case the quantum version admits bound states with negative energies, $\sqrt{H}$ should be replaced by $\sqrt{-H}$

Making use of the equations of motion and deformed algebra relations we arrive at the following expressions :

$$f(x) = \frac{2}{\alpha(H)}\left[\varphi'(x) + g'(x)\sqrt{H}\right]$$

$$f(x)V'(x) - 2f'(x)\left[H - V(x)\right] = \alpha(H)\left\{g(x)\sqrt{H} + \varphi(x) + \phi(H)\right]$$

$$\begin{aligned}\beta \ &= \ 2\sqrt{H}\left[f'(x)g(x) - f(x)g'(x)\right] - \frac{1}{\sqrt{H}}g(x)\left[2f'(x)V(x) + f(x)V'(x)\right] \\ &+ 4f'(x)\phi'(H)\left[H - V(x)\right] - 2f(x)\left[\varphi'(x) + \phi'(H)V'(x)\right]\end{aligned}$$

Construct 2 quantities

$$\mathcal{Q}^{\pm} = A^{\pm} \; e^{\mp \; i \; \alpha(H) \; t}$$

which are time dependent integrals of motion.

Nevertheless, their total time derivative vanishes

$$\frac{d\mathcal{Q}^{\pm}}{dt} \; = \; \{\mathcal{Q}^{\pm} \; , H\} \; + \; \frac{\partial \mathcal{Q}^{\pm}}{\partial t} \; = \; 0$$

Thus $$|\,\mathcal{Q}^{+}\;\mathcal{Q}^{-}\,| = |\,A^{+}\;A^{-}\,|$$

Particular values

$$\mathcal{Q}^{\pm} = c(E)\; e^{\pm\, i\, \theta_0}$$

$$A^{\pm} = c(E)\; e^{\pm\, i\, \{\theta_0 + \alpha(H)\, t\}}$$

where $$c(E) = \sqrt{E - \gamma(H)}$$

θ_0 is determined from initial conditions

For *c(E)* to be real, $E > \gamma(H)$

This condition gives the range of energy values for the classical particle.

The solutions of $\mathcal{Q}^{\pm} = c(E)\, e^{\pm i\, \theta_0}$

gives the trajectories *x(t)* and momenta *p(t)* of the corresponding classical particle in the complex plane.

Explicit examples : exactly solvable models both in quantum & classical versions

- **Classical analogue of *Complex Scarf II potn***

Quantum version displays :

1. Real, discrete spectrum below *PT threshold*, above which complex conjugate pairs of *E*

2. Continuous spectrum admits spectral singularity at the critical point, where *R* and *T* tend to diverge.

Study restricted to bound states only, hence, negative energies (*E* < *0*)

Energy values are continuous for both bound and unbounded motion states

Final expression for *V (x)*

$$V(x) = -\gamma_0 \,\mathrm{sech}^2 \frac{\alpha_0 x}{2} + 2\delta \,\mathrm{sech} \frac{\alpha_0 x}{2} \,\tanh \frac{\alpha_0 x}{2}$$

The parameter δ plays a crucial role

- ***Real δ : Real $V(x)$***
- ***Im δ : PT symmetric $V(x)$***
- ***Complex δ : General Complex $V(x)$***

  (neither PT sym nor η-pseudo-Hermitian)

For

$$V(x) = -\gamma_0 \operatorname{sech}^2 \frac{\alpha_0 x}{2} + 2\delta \operatorname{sech} \frac{\alpha_0 x}{2} \tanh \frac{\alpha_0 x}{2}$$

In the expression

$$A^{\pm} = \mp i f(x) p + \sqrt{H} g(x) + \varphi(x) + \phi(H)$$

$$g(x) \neq 0$$

$$\varphi(x) = 0$$

$$\phi(H) = \frac{\delta}{\sqrt{-H}}$$

$$A^{\pm} = \mp i f(x) p + \sqrt{-H}\, g(x) + \frac{\delta}{\sqrt{-H}}$$

Particular choice

$$g(x) = \sinh \frac{\alpha_0 x}{2}$$

$$f(x) = \cosh \frac{\alpha_0 x}{2}$$

$$\gamma(H) = -\gamma_0 + \frac{\delta^2}{H}$$

so that

$$A^+ A^- = H + \gamma_0 - \frac{\delta^2}{H}$$

$$c(E) = \sqrt{E + \gamma_0 - \frac{\delta^2}{E}}$$

Range of values for *E* as *c(E)* should be real.

Case 1 : δ is real $\delta = \delta_R$

Real Scarf II potential

$$\frac{-\gamma_0 - \sqrt{\gamma_0^2 + 4\delta_R^2}}{2} < E < 0$$

Case 2 : *δ is pure imaginary* $\delta = i\delta_I$

PT symmetric Scarf II potential

$$\frac{-\gamma_0 - \sqrt{\gamma_0^2 - 4\delta_I^2}}{2} < E < \frac{-\gamma_0 + \sqrt{\gamma_0^2 - 4\delta_I^2}}{2}$$

$\gamma_0 \geq |2\delta_I|$ **Real *E* : exact *PT* sym**

$\gamma_0 < |2\delta_I|$ **Complex conj *E* : spon brkn *PT***

Classical system undergoes *phase transition* at

$$\gamma_0 = |2\delta_I|$$

Classical trajectories

$$x(t) = \frac{2}{\alpha_0} \sinh^{-1}\left\{ \frac{1}{E}\left[\delta_R - c_R(E)\sqrt{-E}\cos(\theta_0 + \alpha_0\sqrt{-E}t) \pm c_I(E)\sqrt{-E}\sin(\theta_0 + \alpha_0\sqrt{-E}t)\right]\right\}$$

Classical momenta

$$p(t) = E\left\{ -c_R(E)\sin(\theta_0 + \alpha_0\sqrt{-E}t) \pm c_I(E)\cos(\theta_0 + \alpha_0\sqrt{-E}t) \pm \frac{\delta_I}{\sqrt{-E}} \right\} \times \left[E^2 + \left\{\delta_R - c_R(E)\sqrt{-E}\cos(\theta_0 + \alpha_0\sqrt{-E}t) \pm c_I(E)\sqrt{-E}\sin(\theta_0 + \alpha_0\sqrt{-E}t)\right\}^2\right]^{-1/2}$$

PT symmetric Scarf II potential : $\delta = i\,\delta_I$

Real energy : Unbroken or exact *PT* symmetry

Classical turning points at

$$z_{\pm} = \pm\, a + ib = (z, -z^*)$$

Hence symmetric wrt imaginary axis

For $\alpha_0 = 2$, $\delta = 2i$, $\gamma_0 = 6$, $\quad -0.763932 > E > -5.23607$

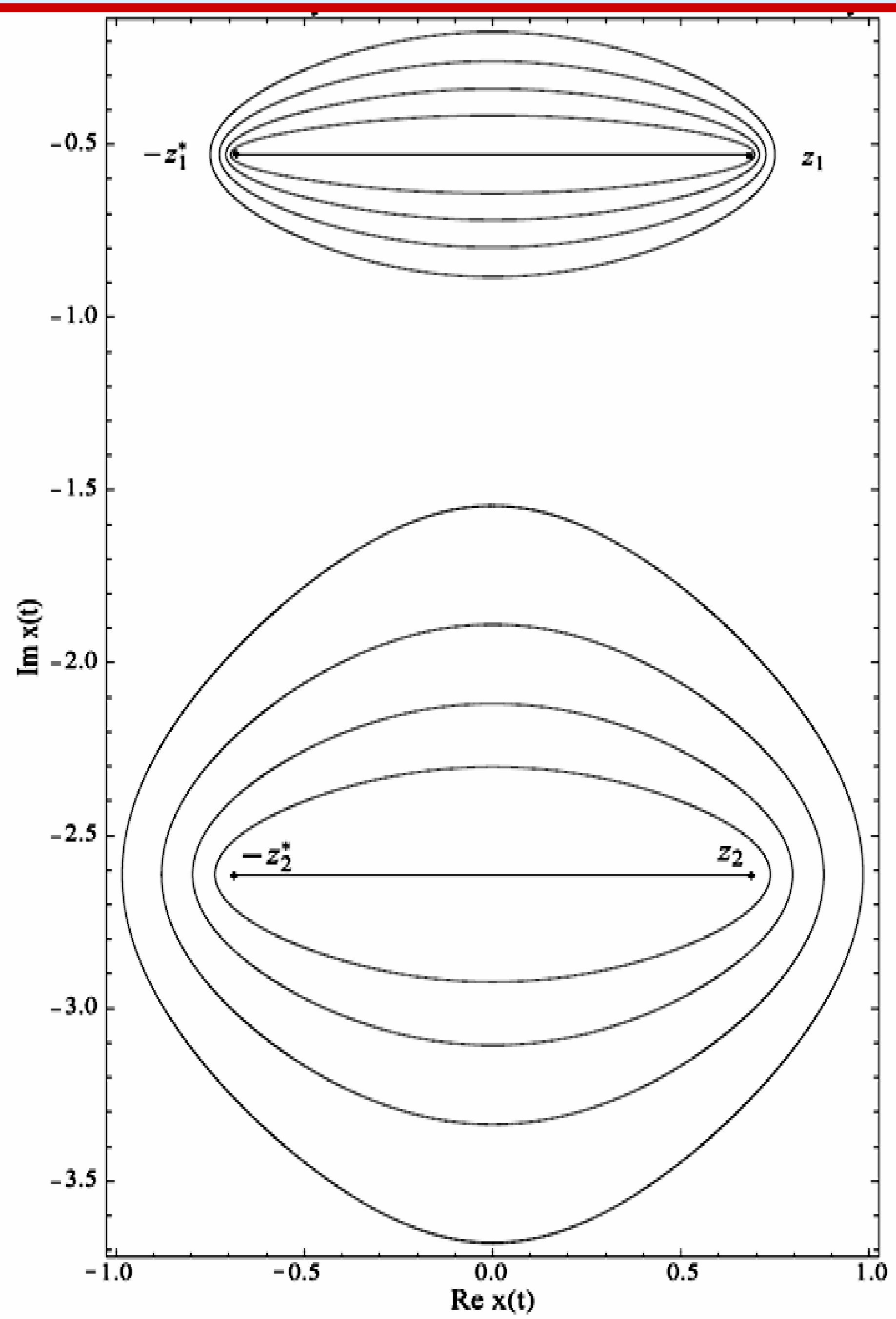
$$E = -3\ ,\ \alpha_0 = 2\ ,\ \delta = 2i\ , \gamma_0 = 6$$

$z = \pm\ 0.781368 - 0.528945\, i$, $\pm\ 0.781368 - 2.61265\, i$, etc

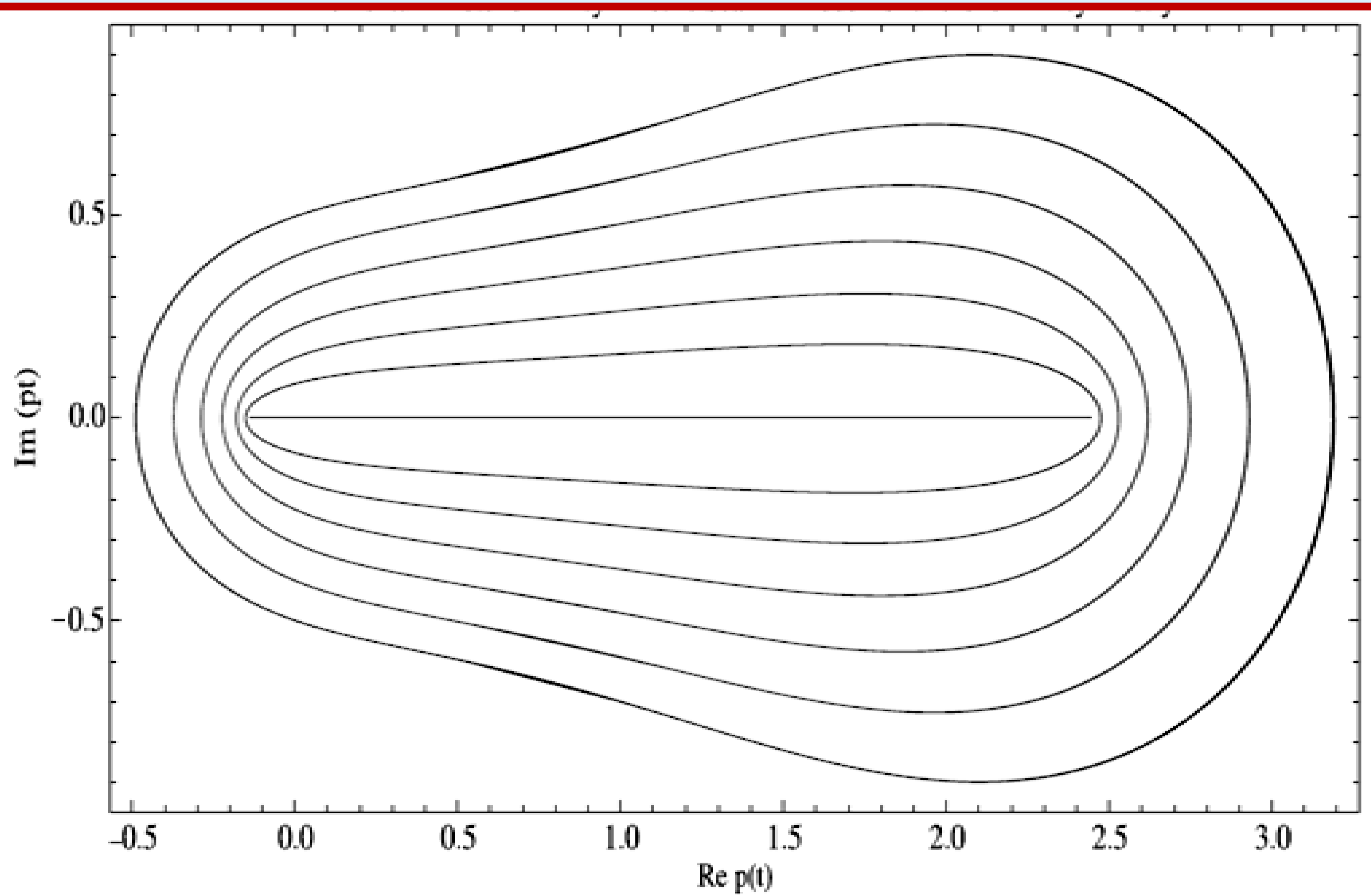
$$(z_1, -z_1^*)\ ,\ (z_2, -z_2^*)$$

Time Period same for each orbit

Plots symmetric wrt im axis



Classical momenta --- plots sym wrt real axis

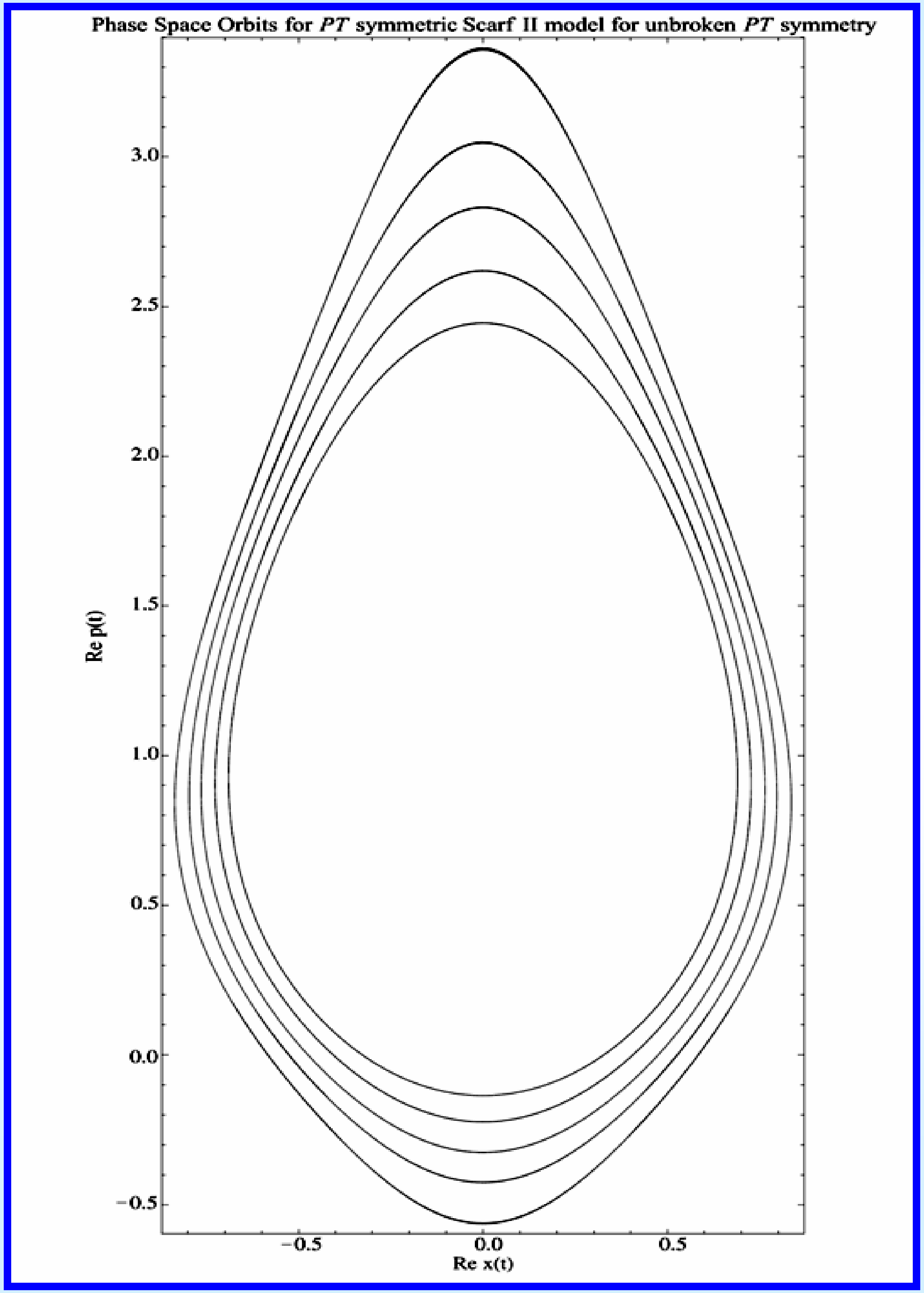


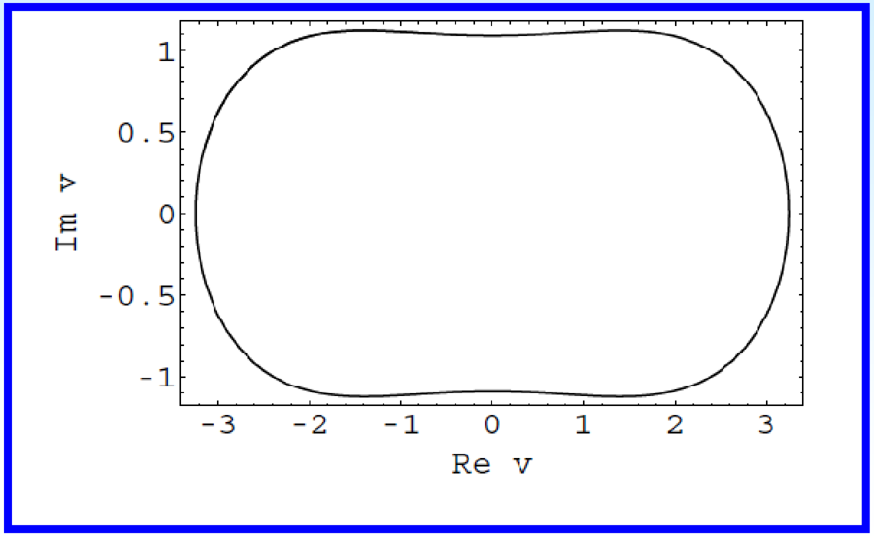
$$E = -3 \ , \ \alpha_0 = 2 \ , \ \delta = 2i \ , \gamma_0 = 6$$

Phase Space Trajectory for unbroken *PT* symmetry

$E = -3, \alpha_0 = 2,$

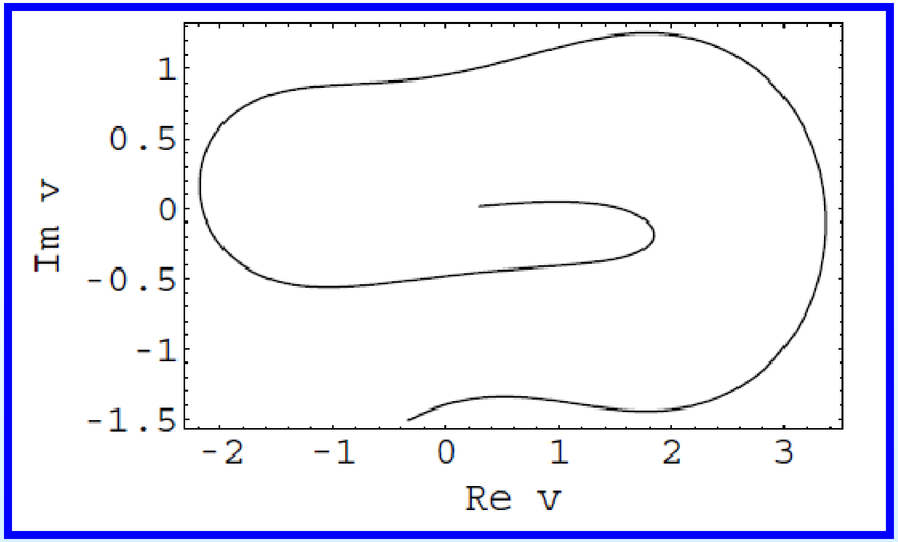
$\delta = 2i, \ \gamma_0 = 6.$





Velocity profile for unbroken *PT* phase

Velocity profile for spontaneously broken *PT* phase



Open Orbits : Spontaneously broken *PT*

$$\gamma_0 < 2 \, | \, \delta_I \, |$$

Turning points no longer of the form ***z, -z****

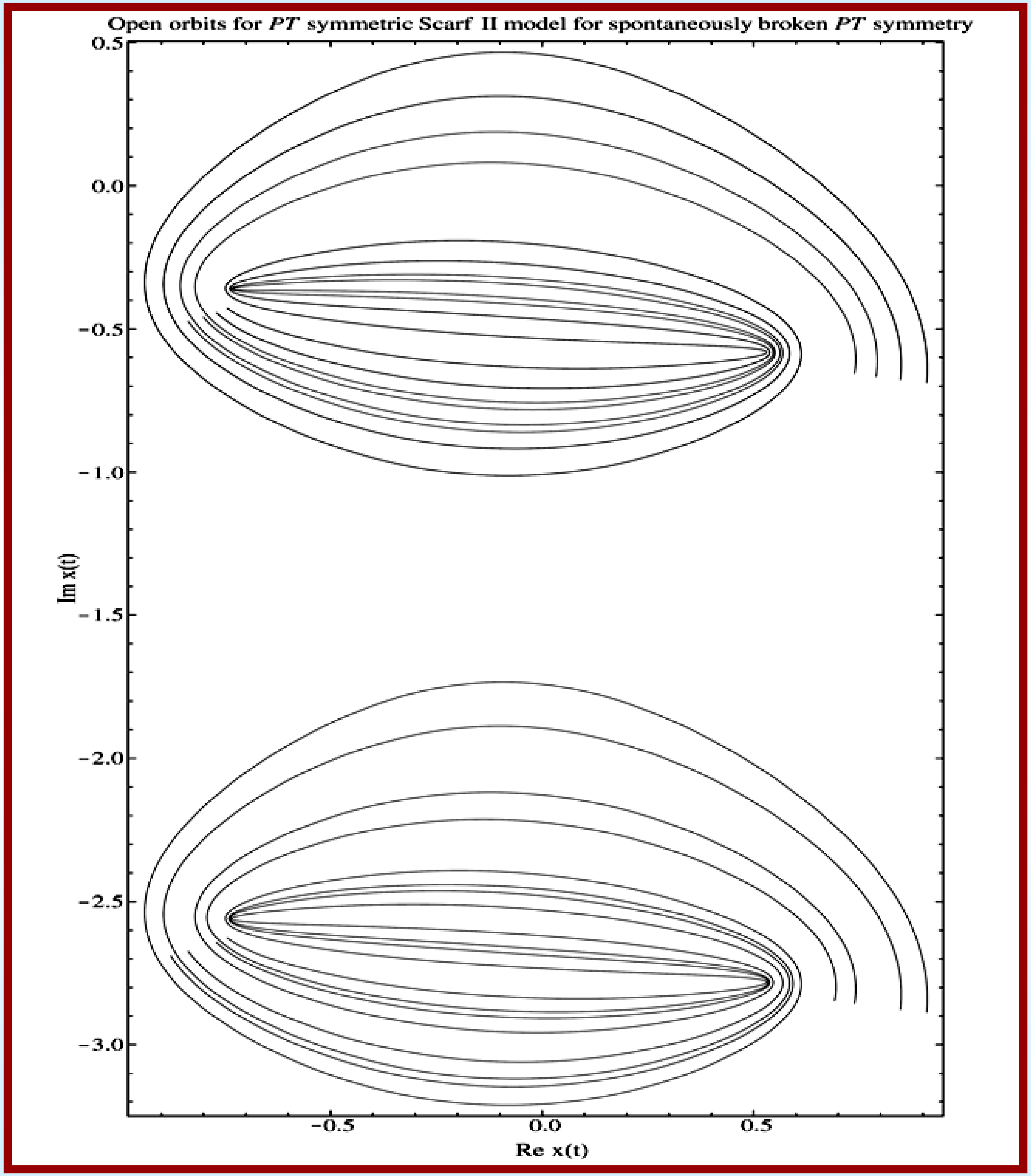
i.e. **no longer symmetric wrt Im axis**

$$x = -0.681374 + 0.5\,i, \quad -0.681374 - 2.64159\,i,$$
$$1.08137 + 0.5\,i, \quad 1.08137 + 3.64159\,i$$

$$\alpha_0 = 2, \; \delta = 2i, \; \gamma_0 = 3$$

E **should lie between** $-1.5 + 1.32288\,i$ *and* $-1.5 - 1.32288\,i$

Plots for $E = -1.5 - 0.3i$

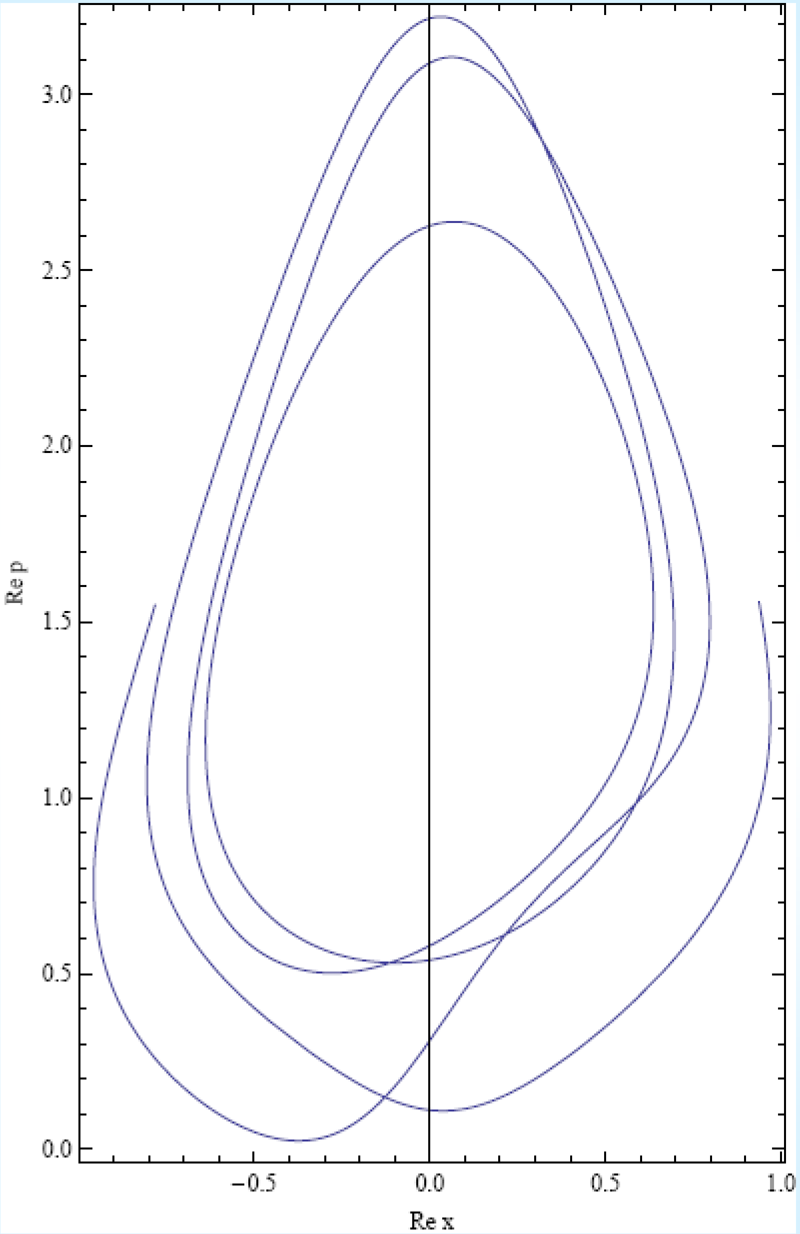

Open orbits for *PT* symmetric Scarf II model for spontaneously broken *PT* symmetry
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2.5
−3.0
Im x(t)
−0.5
0.0
0.5
Re x(t)

Phase Space

Trajectory

for broken

***PT* symmetry**



• Complex Scarf II potential :

$$\delta = \delta_R + i\,\delta_I$$

not η-pseudo Hermitian either

Energies are complex but not complex conjugate pairs

For $\alpha = 2\,,\ \gamma_0 = 3\,,\ \delta = 1 + i$

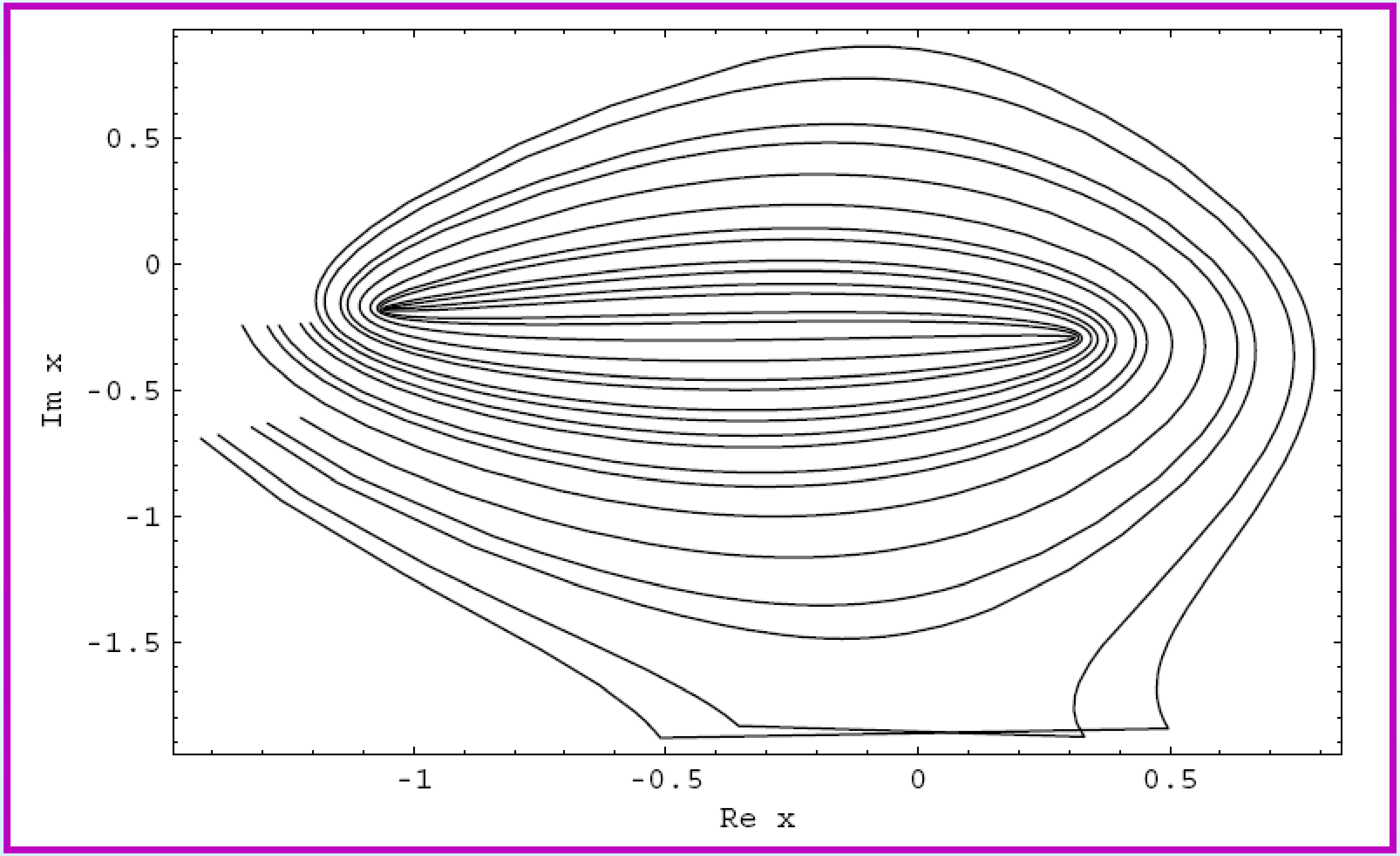
$$-3.12179 - 0.616603i < E < 0$$

For $E = -2 - 0.5\,i$

Classical turning points at

$$z = -1.15046 - 0.227512\,i\,,\quad -0.217307 - 2.82999\,i\,,$$
$$0.217307 - 0.311604\,i\,,\quad 1.15046 - 2.91408\,i\,,\ etc$$

Open orbits :



- **Classical analogue of an η-pseudo Hermitian Quantum Model**

$$V(x) = -v_0 \operatorname{sech}^2(x - \sigma - i\epsilon)$$

Eq of orbit

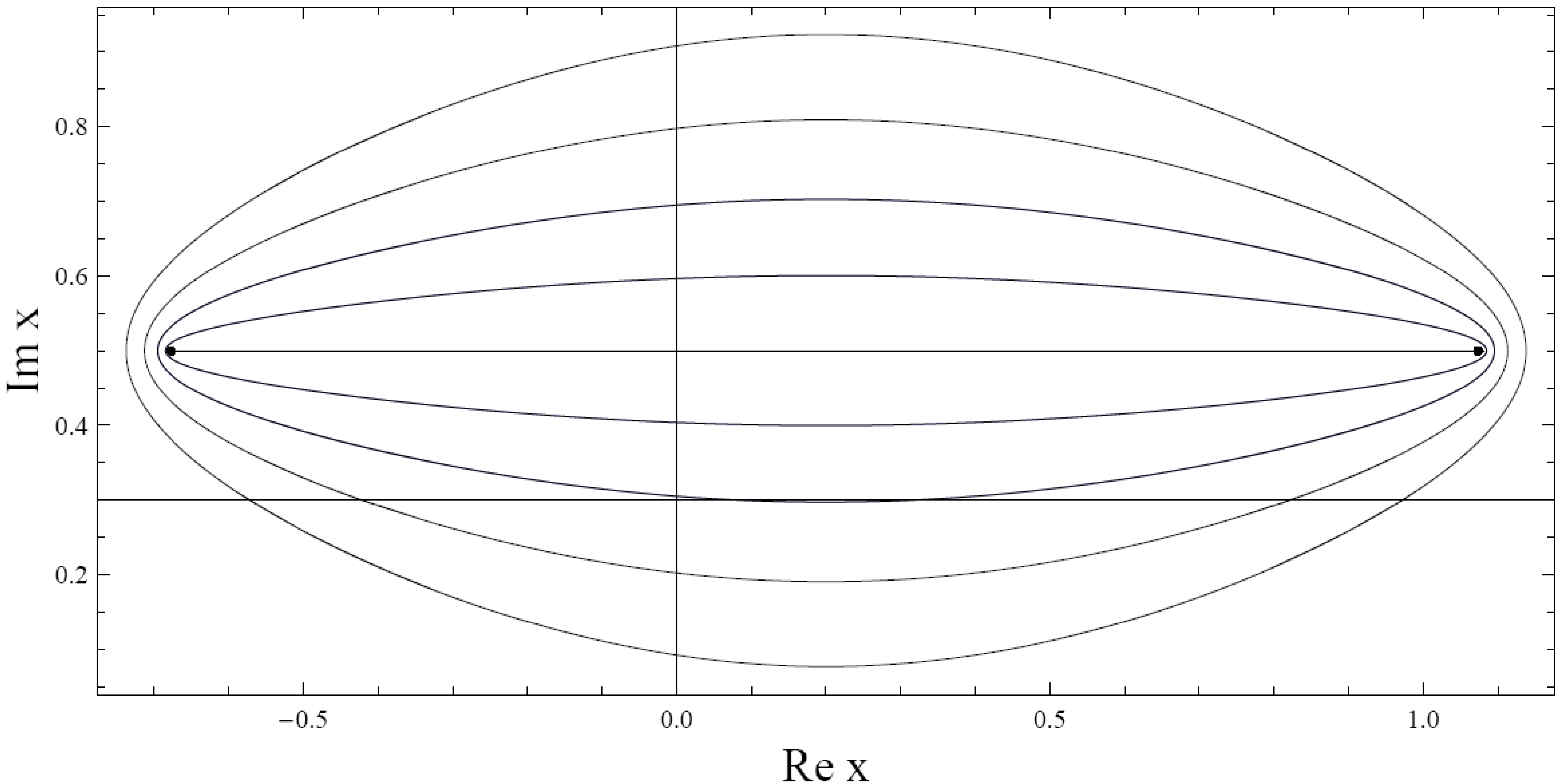
$$x(t) = \frac{2}{\alpha_0}\left[\sigma + i\epsilon + \sinh^{-1}\left(\frac{\sqrt{E+\gamma_0}\cos\left(\theta_0 + \alpha_0\sqrt{-E}t\right)}{\sqrt{-E}}\right)\right]$$

Quantum version shows no abrupt phase transition

classical version shows no irregular behaviour

Model *PT* sym if reflection considered about $x = \sigma$

$\alpha = 2$, $\sigma = 0.2$, $\epsilon = .5, \ \gamma_0 = 6, \ E = -3$



Time Period for each orbit same

CONCLUSIONS :

We have studied exactly solvable classical analogues of some exactly solvable, non Hermitian quantum mechanical Hamiltonians ,

- **with the help of factorization technique**
- **obtained expressions for classical orbit, momenta**
- **plotted the classical orbits, momenta,**

- **Phase Transition from real energies to complex conjugate pairs, in certain non Hermitian *PT* sym systems, is observed in classical systems as well**

For *PT* symmetric systems

- **Below *PT* threshold (Real energy)**
  - **closed trajectories**
  - **same time period for each orbit**
  - **momentum curves closed and regular**
  - **regular phase space trajectories**
  - **orbits sym wrt imaginary axis**

- **Beyond critical point (Complex energy)**
  - **open orbits**
  - **trajectories do not cross**
  - **no sym wrt either real or imaginary axis**

- **For general complex system, without *PT* sym**
  - **trajectories may cross**

Thank You