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# New solutions for antilinear deformations of Coxeter groups with applications to Calogero models

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Dresden, $23^{rd}$ of June 2011

based on collaborations with Andreas Fring

- Operators invariant under antilinear transformations possess real eigenvalues when their eigenfunctions respect this symmetry.
- Non-Hermitian Hamiltonians admitting antilinear symmetry $\mapsto$ used to define consistent quantum mechanical systems with real energy spectra [1]
- $\mathcal{PT}$-symmetry together with a Hamiltonian eg.

$$\mathcal{H}_C(p,q) = \frac{p^2}{2} + \frac{\omega^2}{4} \sum_{\alpha \in \Delta^+} (\alpha \cdot q)^2 + \sum_{\alpha \in \Delta^+} \frac{g_\alpha}{(\alpha \cdot q)^2},$$

  is such a symmetry.
- Focus on antilinear symmetries in general $\mapsto$ usually realized on dynamical variables or fields.

[1] *E. Wigner, Normal form of antiunitary operators, J. Math. Phys. 1 409-13, 1960*

# Today

- Build systematic construction of $\mathcal{PT}$-symmetric deformed root systems
  Based on
  - i *A. Fring and M. Znojil, $\mathcal{PT}$-symmetric deformations of Calogero models, J. Phys. A41, 194010(17)(2008)*
  - ii *A. Fring and M. Smith, Antilinear deformations of Coxeter groups, an application to Calogero models. J. Phys. A43, 325201(28)(2010)*
  - iii *A. Fring and M. Smith, $\mathcal{PT}$ invariant complex $E_8$ root spaces. Int. J. oh Theor. Phys. 50, 974-981 (2011)*
- No solution for some of these maps in ii, extend construction to fill gaps
- Apply deformed root systems to generalized Calogero model

# General mathematical framework

Construct complex extended root systems $\tilde{\Delta}(\varepsilon)$ invariant under a new antilinear involutory map.

- Simple roots : $\alpha_i \in \Delta \subset \mathbb{R}^n \mapsto \tilde{\alpha}_i \in \tilde{\Delta}(\varepsilon) \in \mathbb{R}^n \oplus \imath\mathbb{R}^n$, $\varepsilon \in \mathbb{R}$
- Linear map : $\delta : \Delta \mapsto \tilde{\Delta}(\varepsilon)$
  relating : $\alpha \mapsto \tilde{\alpha} = \theta_\varepsilon \alpha$
- $\tilde{\Delta}(\varepsilon)$ invariant under an antilinear transformation $\omega$
  - **(i)** $\omega : \tilde{\alpha} = \mu_1\alpha_1 + \mu_2\alpha_2 \mapsto \mu_1^*\omega\alpha_1 + \mu_2^*\omega\alpha_2$ for $\mu_1, \mu_2 \in \mathbb{C}$
  - **(ii)** $\omega^2 = \mathbb{I}$
  - **(iii)** $\omega : \tilde{\Delta} \to \tilde{\Delta}$.

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# General mathematical framework

Various candidates for parity transformation:

- Weyl reflections $\sigma_i(x) := x - 2\frac{x\cdot\alpha_i}{{\alpha_i}^2}\alpha_i$ with $1 \leq i \leq \ell$
- Longest element

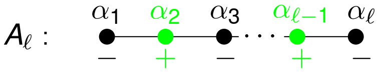
$$\omega_0 = \begin{cases} \sigma^{\frac{h}{2}} & \text{if} \quad \text{h even} \\ \sigma_+\sigma^{\frac{h-1}{2}} & \text{if} \quad \text{h odd} \end{cases}$$

- Factors of Coxeter element $\sigma = \prod_{i=1}^{\ell} \sigma_i$

These candidates together with $\mathcal{T}$ time reversal then constitute the analogue of a $\mathcal{PT}$-operator.
Concentrate on deforming factors of the Coxeter element.

# General mathematical framework

$\sigma_i$ do not generally commute.




Two disjoint sets $V_\pm$ of simple roots $\alpha_i$

$$\sigma := \sigma_- \sigma_+ \quad \text{with} \quad \sigma_\pm := \prod_{i \in V_\pm} \sigma_i \quad \text{with} \quad \sigma_\pm^2 = 1$$

Elements in each set commutes with other elements in the same set.

# General mathematical framework

$\sigma_\pm$ analogue to parity transformations.

$$\sigma_\pm^\varepsilon := \theta_\varepsilon \sigma_\pm \theta_\varepsilon^{-1} = \tau\sigma_\pm : \quad \tilde{\Delta}(\varepsilon) \mapsto \tilde{\Delta}(\varepsilon)$$

And then deformed Coxeter element:

$$\sigma_\varepsilon := \theta_\varepsilon \sigma \theta_\varepsilon^{-1} = \sigma_-^\varepsilon \sigma_+^\varepsilon = \tau\sigma_-\tau\sigma_+ = \tau^2\sigma_-\sigma_+ = \sigma : \quad \tilde{\Delta}(\varepsilon) \mapsto \tilde{\Delta}(\varepsilon)$$

ie. $\sigma_\varepsilon$ acts on $\tilde{\Delta}$ in the same way as $\sigma$ on $\Delta$.
This means $\qquad [\sigma, \theta_\varepsilon] = 0$

# General mathematical framework

- Deformed Coxeter orbits:

$$\Omega_i^\varepsilon := \left\{ \tilde{\gamma}_i, \sigma_\varepsilon \tilde{\gamma}_i, \sigma_\varepsilon^2 \tilde{\gamma}_i, \ldots, \sigma_\varepsilon^{h-1} \tilde{\gamma}_i \right\} = \theta_\varepsilon \Omega_i$$

with $\tilde{\gamma}_i = c_i \tilde{\alpha}_i$, $c_i = \pm$ for $i \in V_\pm$

- Deformed root space:

$$\tilde{\Delta}(\varepsilon) := \bigcup_{i=1}^{\ell} \Omega_i^\varepsilon = \theta_\varepsilon \Delta$$

- invariance:

$$\sigma_\pm^\varepsilon : \tilde{\Delta}(\varepsilon) \to \theta_\varepsilon \sigma_\pm \theta_\varepsilon^{-1} \tilde{\Delta}(\varepsilon) = \theta_\varepsilon \sigma_\pm \Delta = \theta_\varepsilon \Delta = \tilde{\Delta}(\varepsilon)$$

Demand one-to-one relation between individual roots

- limit $\lim_{\varepsilon \to 0}$:

$$\lim_{\varepsilon \to 0} \tilde{\alpha}_i(\varepsilon) = \alpha_i \qquad \lim_{\varepsilon \to 0} \tilde{\Delta}(\varepsilon) = \Delta$$

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# General mathematical framework

Invariant kinetic term in Hamiltonian

$$\alpha_i \cdot \alpha_j = \tilde{\alpha}_i \cdot \tilde{\alpha}_j.$$

$\Rightarrow \theta_\varepsilon$ isometry

$$\theta_\varepsilon^* = \theta_\varepsilon^{-1} \qquad \text{and} \qquad \det \theta_\varepsilon = \pm 1$$

**Summary: properties of $\theta_\varepsilon$**

**(i)** $\theta_\varepsilon^* \sigma_\pm = \sigma_\pm \theta_\varepsilon$

**(ii)** $[\sigma, \theta_\varepsilon] = 0$

**(iii)** $\theta_\varepsilon^* = \theta_\varepsilon^{-1}$

**(iv)** $\det \theta_\varepsilon = \pm 1$

**(v)** $\lim_{\varepsilon \to 0} \theta_\varepsilon = \mathbb{I}$

Solutions?

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Solutions?

# General mathematical framework

$\because [\sigma, \theta_\varepsilon] = 0$, make Ansatz:

$$\theta_\varepsilon = \sum_{k=0}^{h-1} c_k(\varepsilon)\sigma^k, \qquad \lim_{\varepsilon \to 0} c_k(\varepsilon) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}, \quad c_k(\varepsilon) \in \mathbb{C}$$

$\Rightarrow$ into $\theta_\varepsilon^* \sigma_\pm = \sigma_\pm \theta_\varepsilon$ $\quad$ ($c_0 = r_0$, $c_{h/2} = r_{h/2}$, $c_k = \imath r_k$ otherwise)

$$\theta_\varepsilon = \begin{cases} r_0(\varepsilon)\mathbb{I} + \imath \sum\limits_{k=1}^{(h-1)/2} r_k(\varepsilon)(\sigma^k - \sigma^{-k}) & h \text{ odd}, \\ r_0(\varepsilon)\mathbb{I} + r_{h/2}(\varepsilon)\sigma^{h/2} + \imath \sum\limits_{k=1}^{h/2-1} r_k(\varepsilon)(\sigma^k - \sigma^{-k}) & h \text{ even}. \end{cases}$$

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# General mathematical framework

Computing $\theta_\varepsilon$ for all Coxeter groups we find:

- Found closed solutions for $A_{4n-1}, B_{2n}, C_{2n}, D_{2n+1}, E_{6,8}, G_2$.
- No closed formulae for rest of $A_n$
- $B_{2n+1}, C_{2n+1}, D_{2n}, E_7, H_3$ showed that no non trivial solution exists for $\theta_\varepsilon$

Presented in *A. Fring and M. Smith, Antilinear deformations of Coxeter groups, an application to Calogero models. J. Phys. A43, 325201(28)(2010)*

# General mathematical framework

**New modified construction to find solutions where none existed before** $\cdots$
Do this by looking at arbitrary element of the Coxeter group $\tilde{\sigma} = \prod_i \sigma_i$, and following the same procedure as for $\sigma$ with

$$\tilde{\sigma}_\pm := \sigma_\pm \prod_{i \in \tilde{V}_\pm} \sigma_i$$

and since the $\tilde{\sigma}$ is smaller than $\sigma$, $\tilde{h} < h$
which leads to the deformation matrix

$$\tilde{\theta}_\varepsilon = \begin{cases} r_0(\varepsilon)\mathbb{I} + \imath \sum_{k=1}^{(\tilde{h}-1)/2} r_k(\varepsilon)(\tilde{\sigma}^k - \tilde{\sigma}^{-k}) & \tilde{h} \text{ odd}, \\ r_0(\varepsilon)\mathbb{I} + r_{\tilde{h}/2}(\varepsilon)\tilde{\sigma}^{\tilde{h}/2} + \imath \sum_{k=1}^{\tilde{h}/2-1} r_k(\varepsilon)(\tilde{\sigma}^k - \tilde{\sigma}^{-k}) & \tilde{h} \text{ even}. \end{cases} \tag{1}$$

# General mathematical framework

- Now more than just one $\tilde{\sigma}$ for each $\ell$
- Can solve $det(\tilde{\theta}_\varepsilon) = 1$
- $\tilde{\sigma}$ can be divided into different similarity classes $\Sigma_{\tilde{s}}$ characterized by modified exponents $\tilde{s}$ of $\tilde{\sigma}$

$$\tilde{s} = \{\tilde{s}_1^{\lambda_1}, \tilde{s}_2^{\lambda_2}, \cdots, \tilde{s}_{\tilde{h}-1}^{\lambda_{\tilde{h}-1}}, \tilde{s}_{\tilde{h}}^{\lambda_{\tilde{h}}}\}$$

$\tilde{h} = 4$ then $r_1 = \sqrt{r_0^2 - r_0}$ and $r_2 = 1 - r_0$ giving

$$\tilde{\theta}_\varepsilon = r_0(\varepsilon)\mathbb{I} + (1 - r_0(\varepsilon))\tilde{\sigma}^2 + \imath\sqrt{r_0^2(\varepsilon) - r_0(\varepsilon)}(\tilde{\sigma} - \tilde{\sigma}^{-1})$$

For $A_\ell$ the simplest similarity class is

$$\Sigma_{\{1,2,3,4^{\ell-3}\}} = \{\tilde{\sigma}^{(1)}, \cdots, \tilde{\sigma}^{(\ell-2)}\}$$

$$\tilde{\sigma}^{(i)} = \sigma_{i+1}\sigma_i\sigma_{i+2} \quad \text{with} \quad i = 1, \cdots, \ell - 2$$

Can be generalized for $\tilde{h} = 4n$

$$\tilde{\sigma}^{(n,i)} = [(\prod_{k=2}^{n} \sigma_{i-1+4(k-1)}\sigma_{i+1+4(k-1)})\sigma_{i+1}(\prod_{k=1}^{n} \sigma_{i+4(k-1)}\sigma_{i+2+4(k-1)})]$$

Other similarity classes become more complicated such as

$$\Sigma_{\{1,2^2,3,4^{\ell-4}\}} \quad \text{with} \quad \tilde{\sigma}^{(i,j)} = \sigma_i\sigma_{i+2}\sigma_{i+3+j}\sigma_{i+1}$$

# Application to Calogero models

- Undeformed Calogero Hamiltonian

$$\mathcal{H}_C(p,q) = \frac{p^2}{2} + \frac{\omega^2}{4}\sum_{\alpha\in\Delta^+}(\alpha\cdot q)^2 + \sum_{\alpha\in\Delta^+}\frac{g_\alpha}{(\alpha\cdot q)^2},$$

- define the variables

$$z := \prod_{\alpha\in\Delta^+}(\alpha\cdot q) \qquad \text{and} \qquad r^2 := \frac{1}{\hat{h}t_\ell}\sum_{\alpha\in\Delta^+}(\alpha\cdot q)^2,$$

$\hat{h} \equiv$ dual Coxeter number, $t_\ell \equiv \ell$-th symmetrizer of $I$

- Ansatz for solution to $\mathcal{H}_C\psi(q) = E\psi(q)$

$$\psi(q) \to \psi(z,r) = z^{\kappa+1/2}\varphi(r)$$

$\Rightarrow$ solution for $\kappa = 1/2\sqrt{1+4g}$.

$$\varphi_n(r) = c_n \exp\left(-\sqrt{\frac{\hat{h}t_\ell}{2}\frac{\omega}{2}}r^2\right) L_n^a\left(\sqrt{\frac{\hat{h}t_\ell}{2}}\omega r^2\right).$$

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# Application to Calogero models

- eigenenergies

$$E_n = \frac{1}{4}\left[\left(2 + h + h\sqrt{1+4g}\right) l + 8n\right]\sqrt{\frac{\hat{h}t_\ell}{2}}\omega$$

- anyonic exchange factors

$$\psi(q_1, \ldots, q_i, q_j, \ldots q_n) = e^{\imath\pi s}\psi(q_1, \ldots, q_j, q_i, \ldots q_n), \quad \text{for } 1 \leq i, j \leq n,$$

with

$$s = \frac{1}{2} + \frac{1}{2}\sqrt{1+4g}$$

$\because$ $r$ is symmetric and $z$ antisymmetric

# Application to Calogero models

The construction is based on the identities:

$$
\begin{aligned}
\sum_{\alpha,\beta\in\Delta^+} \frac{\alpha\cdot\beta}{(\alpha\cdot q)(\beta\cdot q)} &= \sum_{\alpha\in\Delta^+} \frac{\alpha^2}{(\alpha\cdot q)^2}, \\
\sum_{\alpha,\beta\in\Delta^+} (\alpha\cdot\beta)\frac{(\alpha\cdot q)}{(\beta\cdot q)} &= \frac{\hat{h}h\ell}{2} t_\ell, \\
\sum_{\alpha,\beta\in\Delta^+} (\alpha\cdot\beta)\,(\alpha\cdot q)(\beta\cdot q) &= \hat{h}t_\ell \sum_{\alpha\in\Delta^+} (\alpha\cdot q)^2, \\
\sum_{\alpha\in\Delta^+} \alpha^2 &= \ell\hat{h}t_\ell .
\end{aligned}
$$

Strong evidence on a case-by-case level, but no rigorous proof.

## Application to Calogero models

- antilinearly deformed Calogero Hamiltionian

$$\mathcal{H}_{adC}(p,q) = \frac{p^2}{2} + \frac{\omega^2}{4} \sum_{\tilde{\alpha} \in \tilde{\Delta}^+} (\tilde{\alpha} \cdot q)^2 + \sum_{\tilde{\alpha} \in \Delta^+} \frac{g_{\tilde{\alpha}}}{(\tilde{\alpha} \cdot q)^2}$$

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when identies still hold $\Rightarrow$

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# Application to Calogero models

## Deformed $B_3$-models

- potential $\sum_{\tilde{\alpha}\in\Delta^+} \frac{g_{\tilde{\alpha}}}{(\tilde{\alpha}\cdot q)^2}$ from deformed Coxeter group factors
  $\alpha_1 = \{1, -1, 0\}$, $\alpha_2 = \{0, 1, -1\}$, $\alpha_3 = \{0, 0, 1\}$

$$
\begin{aligned}
\tilde{\alpha}_1 \cdot q &= q_1 - (1 - 2\kappa_0)\, q_2 - 2i\lambda_0 q_3 \\
\tilde{\alpha}_2 \cdot q &= (1 - 2\kappa_0)\,(q_2 - q_3) + 2i\lambda_0\,(q_2 + q_3) \\
\tilde{\alpha}_3 \cdot q &= (1 - 2\kappa_0)\, q_3 - 2i\lambda_0 q_2 \\
\tilde{\alpha}_4 \cdot q &= q_1 - (1 - 2\kappa_0)\, q_3 + 2i\lambda_0 q_2 \\
\tilde{\alpha}_5 \cdot q &= (1 - 2\kappa_0)\,(q_2 + q_3) + 2i\lambda_0\,(q_3 - q_2) \\
\tilde{\alpha}_6 \cdot q &= (1 - 2\kappa_0)\, q_2 + 2i\lambda_0 q_3 \\
\tilde{\alpha}_7 \cdot q &= q_1 + (1 - 2\kappa_0)\, q_2 + 2i\lambda_0 q_3 \\
\tilde{\alpha}_8 \cdot q &= q_1 + (1 - 2\kappa_0)\, q_3 - 2i\lambda_0 q_2 \\
\tilde{\alpha}_9 \cdot q &= q_3
\end{aligned}
$$

# Application to Calogero models

## Deformed $B_3$-models

- potential $\sum_{\tilde{\alpha}\in\Delta^+} \frac{g_{\tilde{\alpha}}}{(\tilde{\alpha}\cdot q)^2}$ from deformed Coxeter group factors
  $\alpha_1 = \{1,-1,0\}$, $\alpha_2 = \{0,1,-1\}$, $\alpha_3 = \{0,0,1\}$

$$
\begin{aligned}
\tilde{\alpha}_1 \cdot q &= q_1 - (1-2\kappa_0)\, q_2 - 2i\lambda_0 q_3 \\
\tilde{\alpha}_2 \cdot q &= (1-2\kappa_0)\,(q_2 - q_3) + 2i\lambda_0\,(q_2 + q_3) \\
\tilde{\alpha}_3 \cdot q &= (1-2\kappa_0)\, q_3 - 2i\lambda_0 q_2 \\
\tilde{\alpha}_4 \cdot q &= q_1 - (1-2\kappa_0)\, q_3 + 2i\lambda_0 q_2 \\
\tilde{\alpha}_5 \cdot q &= (1-2\kappa_0)\,(q_2 + q_3) + 2i\lambda_0\,(q_3 - q_2) \\
\tilde{\alpha}_6 \cdot q &= (1-2\kappa_0)\, q_2 + 2i\lambda_0 q_3 \\
\tilde{\alpha}_7 \cdot q &= q_1 + (1-2\kappa_0)\, q_2 + 2i\lambda_0 q_3 \\
\tilde{\alpha}_8 \cdot q &= q_1 + (1-2\kappa_0)\, q_3 - 2i\lambda_0 q_2 \\
\tilde{\alpha}_9 \cdot q &= q_3
\end{aligned}
$$

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- $\mathcal{PT}$-symmetry for $\tilde{\alpha}$

$$
\begin{aligned}
\sigma_-^{\varepsilon} \quad : \qquad & \tilde{\alpha}_1 \to \tilde{\alpha}_1, \tilde{\alpha}_2 \to \tilde{\alpha}_5, \tilde{\alpha}_3 \to -\tilde{\alpha}_3, \tilde{\alpha}_4 \to \tilde{\alpha}_8, \tilde{\alpha}_5 \to \tilde{\alpha}_2, \\
& \tilde{\alpha}_6 \to \tilde{\alpha}_6, \tilde{\alpha}_7 \to \tilde{\alpha}_7, \tilde{\alpha}_8 \to \tilde{\alpha}_4, \tilde{\alpha}_9 \to \tilde{\alpha}_9, \\
\sigma_+^{\varepsilon} \quad : \qquad & \tilde{\alpha}_1 \to \tilde{\alpha}_4, \tilde{\alpha}_2 \to -\tilde{\alpha}_2, \tilde{\alpha}_3 \to \tilde{\alpha}_6, \tilde{\alpha}_4 \to \tilde{\alpha}_1, \tilde{\alpha}_5 \to \tilde{\alpha}_5, \\
& \tilde{\alpha}_6 \to \tilde{\alpha}_3, \tilde{\alpha}_7 \to \tilde{\alpha}_8, \tilde{\alpha}_8 \to \tilde{\alpha}_7, \tilde{\alpha}_9 \to \tilde{\alpha}_9,
\end{aligned}
$$

- $\mathcal{PT}$-symmetry in dual space

$$
\begin{aligned}
\sigma_-^{\varepsilon} \quad : \qquad & q_1 \to q_1, q_2 \to q_2, q_3 \to -q_3, \imath \to -\imath, \\
\sigma_+^{\varepsilon} \quad : \qquad & q_1 \to q_1, q_2 \to q_3, q_3 \to q_2, \imath \to -\imath.
\end{aligned}
$$

# Conclusions

## Some general conclusions

- We have formed a systematic construction of an antilinear deformation employed as analogue to $\mathcal{PT}$ deformation.
- We have found new models based on this deformation.
- Look at the other algebras.
- Find excited states of the wavefunction $\psi(x)$
- Investigate integrability of the new deformed models.

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**Thank you for your attention**