Models with fundamental length[1] and the finite-dimensional simulations of Big Bang

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purpose

compact presentation of quantum theory of closed "$\mathcal{PT}$" systems

defined via double (H,Θ) [or triple (H,Λ,Θ) etc]

✓ e.g., via Hamiltonian $H \neq H^\dagger$, charge $\Lambda \neq \Lambda^\dagger$, etc

as unitary à la Scholtz et al (1992)

✓ i.e., in *ad hoc* "standard" Hilbert space $\mathcal{H}^{(S)}$

and causal via short-range smearing of coordinates:

✓ MZ, Scattering theory . . . , Phys. Rev. D. 80 (2009) 045009

two fundamental concepts

1. fundamental length θ (= a smearing of Θ)

   - method: lattices, $\dim \mathcal{H}^{(S)} = N < \infty$

   ✓ MZ, ... PT-symmetric chain-models ..., J. Phys. A 40 (2007) 4863

   $\Rrightarrow$ illustrative example I : exactly solvable discrete well

2. horizons $\partial\mathcal{D}$ (= parameter-domain boundaries)

   - physics: quantum catastrophes

   $\Rrightarrow$ benchmark example II : Big-Bang in cosmology

REFERENCES

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the first conceptual innovation: “horizons”

✓ multiples $(H, \Lambda, \ldots, \Theta)$: the “invisible” exceptional points of $\Lambda, \ldots$

MZ, J. Phys. A: Math. Theor. 41 (2008) 244027

models with fundamental length: example I

✓ Chebyshev polynomials: $(H, \Theta)-$formalism

MZ, Phys. Lett. A 375 (2011) 2503

the second conceptual innovation: time-dependent Hilbert spaces

✓ multiples $(H(t), \Lambda(t), \ldots, \Theta(t))$

MZ, "Time-dependent version of cryptohermitian quantum theory", Phys. Rev. D 78 (2008) 085003, arXiv:0809.2874

adiabatic case: example II

✓ "geometry operators" $\Lambda(t)$: quantized gravity

MZ, "Quantum Big Bang . . . ", arXiv:1105.1282

1 the first example (mathematics)

inspiration: ***Hermitian*** discrete square well

Schrödinger equation

$$H^{[U]}\,|\psi_n^{[U]}\rangle = E_n^{[U]}\,|\psi_n^{[U]}\rangle\,, \qquad n = 0, 1, \ldots, N-1$$

N by N Hamiltonian

$$H^{[U]} = \left[\begin{array}{ccccc} 0 & 1 & 0 & \ldots & 0 \\ 1 & 0 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \ldots & 0 & 1 & 0 \end{array}\right] = \left[H^{[U]}\right]^\dagger$$

solvable in terms of Chebyshev polynomials of the second kind

$$|\psi_n^{[U]}\rangle = \begin{bmatrix} U(0, x_n) \\ U(1, x_n) \\ \vdots \\ U(N-1, x_n) \end{bmatrix} ;$$

energies $E_n^{[U]} = 2x_n = \text{real}$

$$E_n^{[U]} = 2 \cos \frac{(n+1)\pi}{N+1} , \qquad n = 0, 1, \ldots, N-1 .$$

today: ***non-Hermitian*** discrete square well

Schrödinger equation

$$H^{[T]}\,|\psi_n^{[T]}\rangle = E_n^{[T]}\,|\psi_n^{[T]}\rangle\,, \qquad n = 0, 1, \ldots, N-1$$

for $\boxed{\text{square-well model of the first kind}}$ with

$$|\psi_n^{[T]}\rangle = \left[\begin{array}{c} T(0, x_n) \\ T(1, x_n) \\ \vdots \\ T(N-1, x_n) \end{array}\right]\,, \qquad E_n^{[T]} = 2\,\cos\frac{(n+1/2)\pi}{N}\,, \quad n = 0, 1, \ldots, N-1$$

and

$$H^{[T]} = \left[\begin{array}{ccccc} 0 & 2 & 0 & \ldots & 0 \\ 1 & 0 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \ldots & 0 & 1 & 0 \end{array}\right] \neq \left[H^{[T]}\right]^\dagger\,.$$

manifest non-Hermiticity

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1. spectrum of $H(\lambda)$ real for $\lambda \in \mathcal{D}^{(H)}$

$$H\,|\psi_n\rangle = E_n\,|\psi_n\rangle$$

2. $\Longrightarrow$ the $\boxed{\text{second}}$ Schrödinger equation

$$H^\dagger\,|\psi_m\rangle\!\rangle = F_m^*\,|\psi_m\rangle\!\rangle\,, \quad \text{i.e.,} \quad \langle\!\langle\psi_m|\,H = F_m\,\langle\!\langle\psi_m|$$

solutions

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1. $\boxed{\text{ket-components}}$

$$\{2|\,\psi^{[T]}\rangle = T(1,x) = x\,,$$

$$\{3|\psi^{[T]}\rangle = T(2,x) = 2\,x^2 - 1\,, \quad ,\ldots\,, \{N|\psi^{[T]}\rangle = T(N-1,x)$$

2. $\boxed{\text{ket-ket-components}}$

$$\{\alpha|\,\psi^{[T]}\rangle\!\rangle = T(n,x)\,, \quad \alpha = 2, 3, \ldots, N\,,$$

3. $\boxed{\text{different}}$:

$$\{1|\psi^{[T]}\rangle = T(0,x) = 1\,, \quad \{1|\psi^{[T]}\rangle\!\rangle = T(0,x)/2 = 1/2\,.$$

the model is cryptohermitian

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choose $\mathcal{H}^{(F)} \equiv \mathbb{C}^N$

and replace the usual inner product

$$(\vec{a},\vec{b}) = \sum_{\alpha=1}^{N} a_\alpha^* b_\alpha$$

by

$$(\vec{a},\vec{b})^{(S)} = \sum_{\alpha=1}^{N}\sum_{\beta=1}^{N} a_\alpha^*\,\Theta_{\alpha,\beta}\,b_\beta$$

this defines $\mathcal{H}^{(S)}$

the THEORY using operator doubles

$$(H(\lambda),\Theta(\kappa))$$

metric

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1. bicompleteness and biorthogonality,

$$I = \sum_{n=0}^{N-1} |\psi_n\rangle \frac{1}{\langle\langle\psi_n|\psi_n\rangle} \langle\langle\psi_n| \,, \qquad \langle\langle\psi_m|\psi_n\rangle = \delta_{m,n} \langle\langle\psi_n|\psi_n\rangle$$

2. formula:

$$\Theta = \sum_{n=0}^{N-1} |\psi_n\rangle\rangle \, |\nu_n|^2 \, \langle\langle\psi_n|$$

3. math: $\Theta > 0$ for $\vec{\nu} \in \triangle^{(\Theta)}$

fundamental-length: band-matrix metrics

solve Dieudonné equation

$$H^\dagger\Theta = \Theta\, H\,, \quad (\Lambda^\dagger\Theta = \Theta\,\Lambda\,, \quad \ldots)$$

1. $\boxed{\text{diagonal metric}}$ = zero-parametric

$$\Theta^{(diagonal)}_{\alpha,\beta} = \delta_{\alpha,\beta}(1-\delta_{\alpha,1}/2)\,\Theta^{(diagonal)}_{N,N}\,, \qquad \alpha,\beta = 1,2,\ldots,N\,.$$

2. $\boxed{\text{tridiagonal}}$ = one-parametric

$$\Theta = K^{(N)}(\lambda) = \left[\begin{array}{cccccc} 1/2 & \lambda & 0 & 0 & \ldots & 0\\ \lambda & 1 & \lambda & 0 & \ddots & \vdots\\ 0 & \lambda & 1 & \ddots & \ddots & 0\\ 0 & 0 & \ddots & \ddots & \lambda & 0\\ \vdots & \ddots & \ddots & \lambda & 1 & \lambda\\ 0 & \ldots & 0 & 0 & \lambda & 1 \end{array}\right]\,.$$

the nontriviality of horizons $\partial\mathcal{D}^{(\Theta)}$

the difficult part is to prove the positivity.

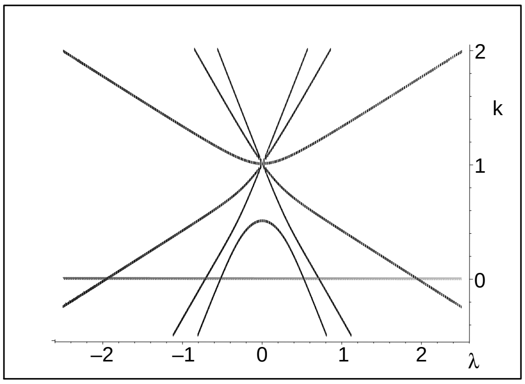

Figure 1: The $\lambda-$dependence of the sextuplet of the eigenvalues of matrix $K^{(6)}(\lambda)$.

(1) matrix $K^{(6)}(\lambda)$ defines the (positive definite) metric $\Theta^{(6)}(\lambda)$

if and only if

$|\lambda| < 0.5176380902 = 2\lambda^{(6)}_{min}$ where $\lambda^{(6)}_{min} = 0.2588190451$ is the smallest positive zero of $T(6,\lambda)$;

(2) matrix $K^{(6)}(\lambda)$ specifies the parity-resembling pseudometric $\mathcal{P}^{(6)}(\lambda)$ (with the three positive and three negative eigenvalues)

if and only if

$|\lambda| > 1.931851653 = 2\lambda^{(6)}_{max}$ where $\lambda^{(6)}_{max} = 0.9659258265$ is the largest zero of $T(6,\lambda)$;

(3) matrix $K^{(6)}(\lambda)$ possesses strictly one negative eigenvalue

if and only if

$2\,\lambda^{(6)}_{min} < |\lambda| < \lambda^{(6)}_{med}$ where $\lambda^{(6)}_{med} = 0.7071067812$ is the third positive zero of $T(6,\lambda)$.

in the limit $\lambda \to 0$:

$k_1(\lambda) \sim 1/2, \quad k_j(\lambda) = 1 + \lambda\, y(\lambda)$

$$-2\,\lambda^5 y^3 - 5\,\lambda^6 y^4 + 3/2\,\lambda^5 y + 6\,\lambda^6 y^2 + 1/2\,y^5 \lambda^5 + y^6 \lambda^6 - \lambda^6 = 0\,,$$

$y_0 \to 0$, $y_{\pm 1} \to \pm 1$, $y_{\pm 2} \to \pm\sqrt{3}$

in the limit $\lambda \to \infty$

star-shaped

p up at $N = 2p$

$N = 6$: $y \approx \pm 1.246979604, \pm 0.4450418679$ and ± 1.801937736

$=$ roots of $U(6, y)$

horizons $\partial\mathcal{D}^{(\Theta)}$ of $K^{(N)}(\lambda)$ at $\boxed{\textbf{any}}$ $N = 2p$:

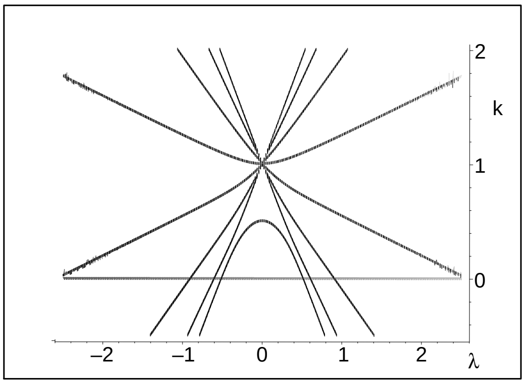

Figure 2: The $\lambda-$dependence of eigenvalues of $K^{(8)}(\lambda)$.

boundary = the smallest roots of $U(2p, \lambda)$,

$$\lambda \in \mathcal{D}^{(\Theta)} = \left(\cos\frac{(p+1)\pi}{2p+1}, \cos\frac{(p)\pi}{2p+1}\right), \qquad N = 2p\,.$$

pentadiagonal Θ

two-parametric

$$\Theta = L^{(N)}(\lambda,\mu) = \left[\begin{array}{cc|cccc|c}
1/2 & \lambda & \mu & 0 & \dots & 0 & 0 \\
\lambda & 1+\mu & \lambda & \mu & 0 & \dots & 0 \\
\hline
\mu & \lambda & 1 & \lambda & \mu & \ddots & \vdots \\
0 & \mu & \lambda & 1 & \ddots & \ddots & 0 \\
0 & 0 & \mu & \ddots & \ddots & \lambda & \mu \\
\vdots & \vdots & \ddots & \ddots & \lambda & 1 & \lambda \\
\hline
0 & 0 & \dots & 0 & \mu & \lambda & 1-\mu
\end{array}\right]$$

degeneracy of vanishing eigenvalues at $\lambda = 0$

$\mu = \sqrt{1 \pm \sqrt{1/2}} = 0.5411961001, 1.306562965.$

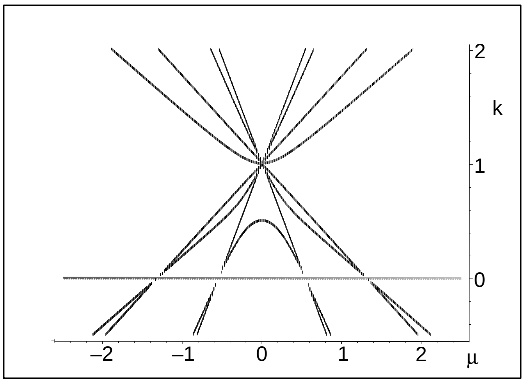

Figure 3: The $\mu-$dependence of the $\lambda = 0$ eigenvalues of matrix $L^{(8)}(\lambda, \mu)$.

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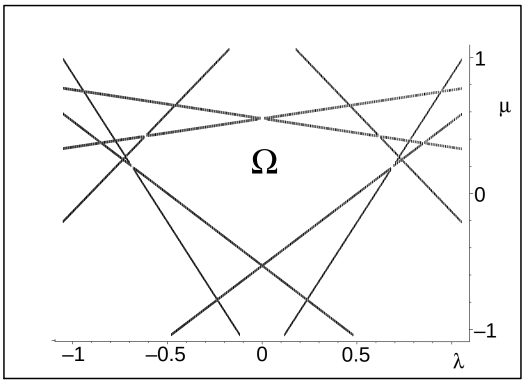

Figure 4: The domain Ω of positivity of matrix $L^{(8)}(\lambda, \mu)$.

secular polynomial seems completely factorizable over reals.

the final square-well message

the dynamics is well controlled by the variations of the set κ of the optional parameters in the metric operator $\Theta = \Theta(\kappa)$

in particular, this may change the EP horizons via Λ

in our example: schematic model

made compatible with the standard postulates of quantum theory

each multiindex $\vec{\kappa} \in \mathcal{D}^\Theta$ numbers the respective Hilbert spaces

the summary of introduction

cryptohermitian discrete square well

✓ elementary

cryptohermiticity (= hidden Hermiticity)

fundamental length (short-ranged Θs)

no EPs ($\partial\mathcal{D} = \emptyset$)

✓ simplest $H \neq H^\dagger$

non-unitary Dyson hermitizer $\Omega : \mathcal{H}^{(F)} \ \to \ \mathcal{H}^{(P)} \ \sim \ \mathcal{H}^{(S)}$

energies = real, explicit

✓ non-numerical at any N:

short-range, band-matrix Θ

EP horizons via additional Λs (cf. the second example)

2 the second example (physics):

a toy-model quantum Universe
in the vicinity of Big Bang

arXiv:1105.1282

”10th Workshop on Quantization, Dualities and Integrable Systems”

(April 22 - 24, 2011), invited talk

the model (H, Λ, Θ)

1. purely kinetic generator of time evolution ("Hamiltonian")

$$H = H^{(N)} = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \ddots & \vdots & \vdots \\ 0 & -1 & 2 & \ddots & 0 & 0 \\ 0 & 0 & \ddots & \ddots & -1 & 0 \\ \vdots & \vdots & \ddots & -1 & 2 & -1 \\ 0 & 0 & \dots & 0 & -1 & 2 \end{bmatrix}$$

2. N spatial grid points $g_j(t)$ treated as eigenvalues of "space geometry"

$$\Lambda = \hat{G} = \hat{G}^{(N)}(t) = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \dots & \gamma_{1N} \\ \gamma_{21} & \gamma_{22} & \dots & \gamma_{2N} \\ \dots & \dots & \dots & \dots \\ \gamma_{N1} & \gamma_{N2} & \dots & \gamma_{NN} \end{bmatrix} .$$

nothing before Big Bang and after Big Crunch

1. N conditions of reality of the spectrum

$$\text{Im}\; g_j(t) = 0\,, \qquad j = 1, 2, \ldots, N\,, \qquad t_{initial} \le t \le t_{final}\,.$$

2. the partial *or* complete absence of measurability of the space,

$$\text{Im}\; g_j(t) \neq 0\,, \qquad j = 1, 2, \ldots, N_{BBC}\,, \qquad t \notin [t_{initial}, t_{final}]\,, \qquad N_{BBC} \le N\,.$$

3. $\boxed{\text{BBC phenomenon}}$ simulated by $N-1$ conditions

$$\lim_{t \to t_c} g_j(t) = g_N(t_c) := g_c\,, \qquad j = 1, 2, \ldots, N-1$$

of a complete confluence of the $N-$plet of eigenvalues.

4. $N = 4$ illustration: p.t.o.

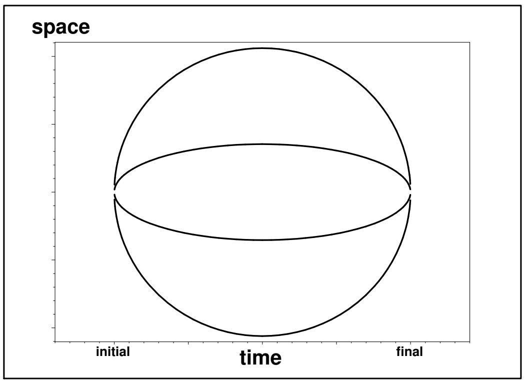

Figure 5: Quantized geography-history of the four-grid-point Universe.

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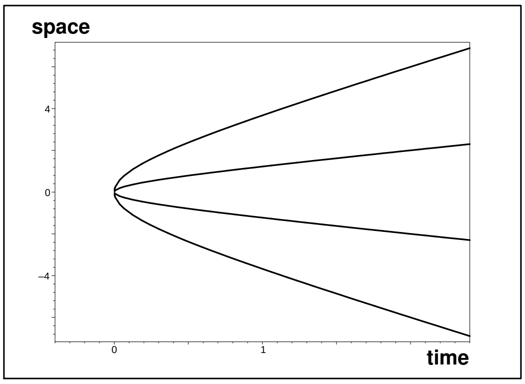

Figure 6: The open-universe alternative

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mathematics

1. the necessary non-Hermiticity in $\mathcal{H}_{\Theta=I}^{(first)}$,

$$\hat{G}(t) \neq \hat{G}^{\dagger}(t)$$

2. the sufficient (crypto)hermiticity in $\mathcal{H}_{\Theta\neq I}^{(second)}$,

$$H = H^{\ddagger} := \Theta^{-1}\, H^{\dagger}\, \Theta \;\equiv\; \Theta^{-1}\, H\, \Theta\,,$$

$$\hat{G}(t) = \hat{G}^{\ddagger}(t) := \Theta^{-1}\, \hat{G}^{\dagger}(t)\, \Theta\,.$$

3. the ansatz

$$\Theta(t) = a(t)\, I + b(t)\, H + c(t)\, H^2 + \ldots + z(t)\, H^{N-1}$$

4. the reduced Dieudonné equation (using $[A, B]_{\dagger} := AB - B^{\dagger}A$),

$$a(t)\, [I, \hat{G}(t)]_{\dagger} + b(t)\, [H, \hat{G}(t)]_{\dagger} + c(t)\, [H^2, \hat{G}(t)]_{\dagger} + \ldots + z(t)\, [H^{N-1}, \hat{G}(t)]_{\dagger} = 0\,.$$

the alternative to Penrose's scenario

teaching by example: $N = 2$

1. take three real parameters with positive $r(t) > 0$ and two eigenvalues,

$$\hat{G}^{(2)}(t) = \left(\begin{array}{cc} -r(t) & -v(t) \\ u(t) & r(t) \end{array} \right), \quad g_{\pm}^{(2)}(t) = \pm\sqrt{r^2(t) - u(t)v(t)}$$

2. essentially one-parametric via a re-parametrization,

$$u(t) = \frac{1}{2}\varrho(t)\, e^{\mu(t)}\,, \qquad v(t) = \frac{1}{2}\varrho(t)\, e^{-\mu(t)}$$

3. conclude: $\varrho(t) =$ "proper time" and the system is

   unobservable at $\varrho(t) < -2r(t)$,

   observable at $-2r(t) \le \varrho(t) \le 2r(t)$

   unobservable again at $\varrho(t) > 2r(t)$.

metric $\Theta^{(N)}$

1. two-free-parameters anzatz,

$$\Theta^{(2)}(t) = \left(\begin{array}{cc} a(t)+2b(t) & -b(t) \\ -b(t) & a(t)+2b(t) \end{array} \right)\,, \qquad H^{(2)} = \left(\begin{array}{cc} 2 & -1 \\ -1 & 2 \end{array} \right)$$

2. the positivity of eigenvalues $\theta_\pm(t) = a(t) + 2b(t) \pm b(t)$

   = the single constraint $a(t) > \max[-b(t), -3b(t)]$

3. Dieudonné conditions = another single constraint,

$$2b(t)r(t) + u(t)a(t) + 2b(t)u(t) + v(t)a(t) + 2b(t)v(t) = 0\,.$$

4. metric, finally,

$$\Theta^{(2)}(t) = \left(\begin{array}{cc} 2r(t) & u(t)+v(t) \\ u(t)+v(t) & 2r(t) \end{array} \right)\,.$$

discussion

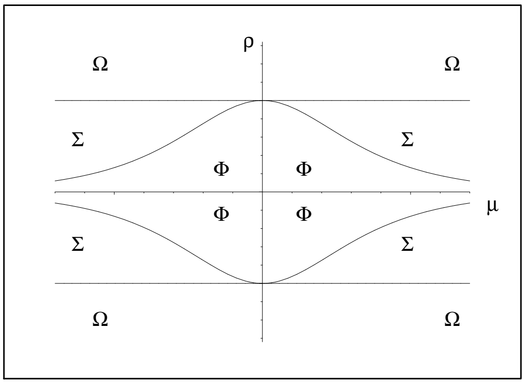

Figure 7: Physical domain (marked by Φ) in $(\rho, \mu)-$plane.

3 outlook

the "three-Hilbert-space quantum theory":

0. textbook level quantum theory :
prohibitively complicated Hamiltonian $\mathfrak{h}$
generating unitary time evolution
P: **physics** = **trivial**
calculations = practically impossible

simplification

unitary equivalence

1. the same state ψ represented
in the **false** Hilbert space
F: calculations = **feasible**
physical meaning = lost

hermitization $\longrightarrow$

2. amended inner product
standardized representation
S: picture = **synthesis**
physics = reinstalled

the sense of PHHQP/THSQT

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✓ one of the most remarkable features of quantum mechanics may be seen in the robust nature of its “first principles” (here, not violated)

✓ probabilistic interpretation practically did not change during the last cca eighty years (here, not violated)

✓ in contrast, innovations do not seem to have ever slowed down:

here, non-unitary Fourier (a.k.a. Dyson) transformation Ω

✓ the first new physics behind PHHQP/THSQT: see Scholtz et al (1992):

fermions = $\Omega \times$ bosons

- summarizing the message: the *dynamical* content of phenomenological quantum models may/should be encoded not only in the Hamiltonian (and other observables) but also, equally efficiently, in metric operators Θ