On the Path Integral for Non–Commutative (NC) Theories

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Outline of Talk

I Motivation for QFT on non-commutative (NC) spacetime

II Particular set–up of QFT on NC spacetime

III Path integral (Hamiltonian approach):
a) in time/space NC QFT corresponding to $T^\star$-ordering
b) in time/space NC QFT corresponding to T-ordering

IV Summary & outlook

I Motivation for QFT on NC spacetime

- 1)Snyder 1947, Yang 1947:
  Hope to remove divergences by introducing a minimum length
- 2)Gedankenexperiment:
  (Doplicher, Fredenhagen, Roberts: [DFR 1995])
  Creation of micro–black holes in scattering events with high energy transfer restricts possible resolution of spacetime events. Below Planck scale, measurements become meaningless.
  $\Longrightarrow$ spacetime uncertainty relations
- 3)String theory:
  (Connes, Douglas, Schwarz; Schomerus ; Seiberg, Witten (1998/ 1999))
  Low energy limit of open string attached to a D–brane in a constant background magnetic field can be desribed by QFT on NC *space* (, not NC *spacetime*).

II Particular set–up of QFT on NC spacetime (1)

Popular idea to implement non–commutative structure :

• Use Weyl–Moyal correspondence & replace the product of functions on commutative spacetime by Moyal–product ($*$–product):

$$(f_1 * f_2)(x) := \left[\exp(\tfrac{i}{2}\theta^{\mu\nu}\partial_\mu^x\partial_\nu^y)f_1(x)f_2(y)\right]_{y=x}$$

• $[\hat{x}_\mu, \hat{x}_\nu] =: i\theta_{\mu\nu}1$; $\hat{x}_\mu$, $\hat{x}_\nu$: coordinate operators; $\theta_{\mu\nu}$: real, antisymmetric, constant matrix ($d = 1 + 3$)

• trace property: $\int d^4xf * g(x) = \int d^4xf \cdot g(x)$ for $f, g \,\epsilon\, \mathcal{S}(\mathbb{R}^{3+1})$

(• here: time/space non–commutativity ($\theta^{0i} \neq 0$, $i = 1, 2, 3$))

II Particular set–up of QFT on NC spacetime (2)

Starting point for QFT on NC spacetime

- Ansatz for free (neutral massive scalar) theory:

$$S^{NC}_{kin} = \frac{1}{2} \int d^4x : (\frac{\partial}{\partial t}\phi * \frac{\partial}{\partial t}\phi)(x) : + \\ : (\partial_i \phi * \partial^i \phi)(x) : + m^2 : (\phi * \phi)(x) : = S_{kin}$$

due to trace property for star product

$\Longrightarrow$ free QFT in NC case equals free (ordinary) QFT

- Ansatz for interaction theory:

$$S^{NC}_{I} \propto \frac{1}{2} \int d^4x \lambda : (\phi * \ldots * \phi)(x) : \propto \\ \lambda \int d^4k_1 \ldots \int d^4k_n : \check{\phi}(k_1) \ldots \check{\phi}(k_n) : e^{\frac{-i}{2} \sum\limits_{i<j} k_i^\mu \theta_{\mu\nu} k_j^\nu} \delta^4(\sum k_i)$$

$\Longrightarrow$ Perturbation theory (generally): Vertices contain trigonometric functions of momenta (twisting factor)

IIIa Path Integral corresponding to $T^\star$–Ordering (1)

- Naïve ansatz for path integral ($\theta_{0i} \neq 0$):

Choose *nonlocal* interaction: $\mathcal{L}_{int}(\phi)_* := \phi * \phi * \phi(x)$, e. g., and plug it in the formula for generating functional of *local* case ($\Delta_c(z)$: causal propagator):

$$Z[J] = N_{00} \exp[i \int d^4 z \mathcal{L}_{int}(\frac{\delta}{i\delta J(z)})_*] \times \\ \times \exp[\frac{-1}{2} \int d^4 a \int d^4 b J(a) \Delta_c(a-b) J(b)]$$

- Perturbative expansion leads to naïve Feynman rules: Graphs with causal propagators as internel lines, only difference: every vertex is multiplied by a factor (*trigonometric function* of momenta)

- Example (fishgraph):

$$\int \frac{d^4 q}{(2\pi)^4} \frac{i}{q^2 - m^2 + i\epsilon} \cos^2(\frac{p_\mu \theta^{\mu\nu} q_\nu}{2}) \frac{i}{(p-q)^2 - m^2 + i\epsilon}$$

IIIa Path Integral corresponding to $T^\star$–Ordering (2)

Further remarks:

- These Feynman rules are also obtained by starting from Gell–Mann - Low formula (canonical approach) and applying $T^\star$–operator ($T^\star$–product: all time derivatives of star product act after time ordering, see Heslop & Sibold [11/04])
- According to Gomis & Mehen [02,00]: Feynman rules violate unitarity.
- Feynman rules violate causal time–ordering (C.D., in preparation).
- No loss of (manifest) covariance (p. c. with Prof. Fredenhagen.).
- See K. Fujikawa [06/04]: same p. i. from e. o. m.: same Feynman rules as above

IIIb Path Integral corresponding to T–Ordering (1)

- Question: In the case of $\theta_{0i} \neq 0$, can one modify the generating functional (of the local theory) in such a way that the Feynman rules preserve *causal time ordering*?

- Answer: Yes!

$$Z[J] = N_{00} \exp[i \int d^4 z [\mathcal{L}_{int}(\frac{\delta}{i\delta J(z)})_*]_{\vec{\theta}}] \times$$
$$\times \exp[\frac{-1}{2} \int d^4 a \int d^4 b J(a) T\Delta_+(a-b) J(b)]$$

- $T\Delta_\pm(z) := \vartheta(z^0)\Delta_+(z) + \vartheta(-z^0)\Delta_+(-z) = \Delta_c(z)$

- $[(\frac{\delta}{\delta J(x)})_*]_{\vec{\theta}}$: For each time–ordered configuration take first the time derivative (associated to θ_{0i}) of $\Delta_+(x)$. Then, realize the time ordering by multiplication with step function. (The argument of the step function never contains θ_{0i}).

IIIb Path Integral corresponding to T–Ordering (2)

Main result: Feynman rules are the same as those derived within the canonical approach and leading to old–fashioned perturbation theory (OTO) (equivalence between canonical approach and path integral)

• Old–fashioned perturbation theory (OTO) (Liao & Sibold [05/02], [06/02]; see also Liao & Dehne [11/02]): Start from Gell–Mann - Low formula and apply T–ordering.

• T–product: all time derivatives of star product act before time ordering is applied (See also Fujikawa [04/06], [04/10]; Heslop & Sibold [11/04].)

• These Feynman rules maintain unitarity and causal time–ordering.

• Loss of (manifest) covariance.

• Derivation of Feynman rules from generating functional is less tedious than in the canonical case!

IIIb Path Integral corresponding to T–Ordering (3)

Excursion: Time–ordered perturbation theory adapted to NC field theory (OTO), fishgraph example:

$$\mathcal{T} \propto \lambda^2 \sum_{\sigma_{1,2}\epsilon\{-,+\}} \int \frac{d^3p_1}{\omega_{\vec{p}_1}} \int \frac{d^3p_2}{\omega_{\vec{p}_2}} \frac{1}{4}\left(1+\sigma_1 \frac{k_1^0}{\omega_{\vec{k}_1}}\right)\left(1+\sigma_2 \frac{k_2^0}{\omega_{\vec{k}_2}}\right)$$

$$\cdot \quad 2\pi\delta(k_1^0-k_2^0)\delta^3(\vec{p}_1+\vec{p}_2-\vec{k}_1)\delta^3(\vec{p}_1+\vec{p}_2-\vec{k}_2)$$

$$\cdot \quad \left(\frac{(\sum_{sym} e^{-i(-k_{1,\sigma_1},p_{1+},p_{2+})} e^{-i(-k_{2,\sigma_2},p_{1+},p_{2+})})}{k_1^0-\omega_{\vec{p}_1}-\omega_{\vec{p}_2}+i\epsilon} \right.$$

$$+ \quad \left. \frac{(\sum_{sym} e^{-i(-k_{1,\sigma 1},p_{1-},p_{2-})} e^{-i(-k_{2,\sigma 2},p_{1-},p_{2-})})}{-k_2^0-\omega_{\vec{p}_1}-\omega_{\vec{p}_2}+i\epsilon} \right)$$

- $\omega_{\vec{p}} := \sqrt{m^2+\vec{p}^2}$, $(a,b,c) := a\wedge b + a\wedge c + b\wedge c$, $a\wedge b := \frac{a_\mu \theta^{\mu\nu} b_\nu}{2}$, $p_\sigma := (\sigma\omega_{\vec{p}}, \vec{p})^\tau$
- Feynman graph decomposes into a part with retardation property and advancement property
- Feynman rules corresponding to T–ordering (OTO): four–momenta in the NC phase are on–shell (loss of covariance)
- Just compare to Feynman rules corresponding to $T^\star$–ordering: four–momenta are off–shell (covariance)

IV Summary and outlook

Summary and outlook

• Main result: successful derivation of path integral formula corresponding to the T–product in canonical case (Hamiltonian approach)

• Feynman rules are identical to those of OTO and thus preserve unitarity and causal time–ordering.

• time ordering (or rather quantization prescription) not rigidly implemented in the path integral

• in progress: path integral based on T–operator
I) in u–coordinates, II) starting from field equation

• in progress: Wick rotation and Euclidean Feynman rules (reflection positivity:$\theta_{0k} \to \pm i\theta_{0k}$ $(k = 1, 2, 3)$, effect on famous UV/IR–connection (!))

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