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Critical phenomena in atomic physics

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Abstract

It is shown that the threshold energy $\varepsilon = 0$ for complete break-up of N charged particles represents a fixed point of the dynamical system. Renormalization theory akin to thermodynamical phase transitions is used to determine the exponent β of the power law for the fragmentation cross section $\sigma(\varepsilon) \propto \varepsilon^{\beta}$ near the fixed point. In the generic case β is the ratio of two rate constants expressed in terms of Liapunov exponents which emerge from the stability analysis of the critical point. Using the derived low and the well-known high-energy behavior, a universal shape function for direct ionization by charged particles is given. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

The threshold cross section for the fragmentation of N charged particles is related to a critical point of the N-body Hamiltonian. Scaling and renormalization techniques are used to derive the (singular) energy dependence of the threshold cross section constituting the threshold law $\sigma(\varepsilon \to 0) \propto \varepsilon^{\beta}$. Here, $\varepsilon = (E - E_c)/(E_c + E_0)$ measures the distance from the threshold energy E_c in an arbitrary energy scale E_0 and the exponent β will turn out to be the ratio of two rates determined by the local Liapunov exponents of the critical (fixed) point. We will illustrate the abstract derivation by the example of electron impact ionization of a hydrogen-like atom or ion for which already Wannier [1] calculated the threshold behavior in 1953. Finally, we will demonstrate that threshold behavior is useful even for energies much higher than E_c through the introduction of a universal shape function for the ionization cross section which only depends on the threshold exponent β.

2. The multi-fragmentation threshold as a critical point

Let us define a dynamical system by the Hamiltonian $H = K + V$, describing the interaction V of N particles with kinetic energy K, expressed in Jacobi coordinates as

$$K = \sum_{i}^{N-1} \frac{\mathscr{P}_i^2}{2m_i}, \tag{1}$$

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where m_i is the reduced mass along the Jacobi coordinates $\mathscr{R}_i$ with conjugated momenta $\mathscr{P}_i$. We will work in mass-scaled-hyperspherical coordinates from now on. In these coordinates only one length is specified, the radius of the hypersphere of all particles. The hyperradius R is defined by the Euclidean distance $R^2 = \sum_i R_i^2$ of all mass weighted-Jacobi vectors $\boldsymbol{R}_i = \sqrt{m_i/m_0}\mathscr{R}_i$, where m_0 is an arbitrary mass scale which we set to unity since we work in atomic units unless stated otherwise. All other coordinates are angles, abbreviated as $\boldsymbol{\omega} = (\alpha_1, \dots, \alpha_{N-2}, \theta_1, \dots, \theta_{2N-2})$, where θ_i is the geometrical angle describing the relative position of the $\boldsymbol{R}_j$ in space, while the so-called hyperangles fix the lengths $R_i = |\boldsymbol{R}_i|$ relative to each other, e.g. $\alpha_1 = \arctan R_1/R_2$ [2].

The potential is a function of all coordinates $V = V(R, \boldsymbol{\omega})$ of which we only require that

$$V(R \to \infty, \boldsymbol{\omega}_0) \to 0 \tag{2}$$

for $\boldsymbol{\omega} = \boldsymbol{\omega}_0$ fixed. This limit represents the fully fragmented N-body system where all particles are force free and infinitely far separated from each other. To simplify the derivation of the threshold law we will only consider full fragmentation specified by Eq. (2). The result can be easily generalized to partial fragmentation into less than N fragments at the respective threshold energy. Naturally, as defined by Eq. (2), the threshold energy for full fragmentation is $E_c = 0$ and the dimensionless energy takes the form $\varepsilon = E/E_0$.

The idea for the theoretical description of threshold fragmentation as a critical phenomenon is based on the following observation: If the N-particle system has completely fragmented, its potential energy approaches zero according to Eq. (2). Close to threshold $E \approx 0$ this implies that the kinetic energy will be small too, since we have $0 \approx E = K + V \approx K$. The almost vanishing kinetic energy after threshold fragmentation has two important consequences which lead directly to the description of threshold fragmentation as a critical phenomenon.

(i) With $K(E \to 0) \to 0$ threshold fragmentation will proceed with the minimum of kinetic energy required to fragment the system. The signature of (complete) fragmentation is that the hyperradius grows, $R \to \infty$ and this requires a finite velocity $\dot{R} \neq 0$. On the other hand, all other (angular) velocities may vanish $\dot{\omega}_i = 0$ and the corresponding degrees of freedom do not have kinetic energy. Preserving this scenario over a finite interval of time, requires a fixed point condition for all angles with the intuitive picture of the N-body system being frozen in a certain geometrical shape while its size is continuously growing with increasing R. Note that the geometrical shape refers to the $3N-3$ dimensional hyperspace and includes the geometrical positions of the N particles in the three-dimensional physical space as well as the ratio of the mutual distances between the particles which remains the same.

(ii) The almost vanishing kinetic energy near threshold implies also that the fragments move asymptotically (for $R \to \infty$) infinitely slowly.

Property (ii) suggests that it might be possible to formulate the dynamical process of threshold fragmentation (which requires also $t \to \infty$) by renormalization of the time evolution of the phase space variables about a fixed point. The phase space variables of the Hamiltonian then play the same role as the coupling constants in the theory of traditional phase transitions, see e.g. Ref. [3]. First, we must find the relevant fixed point. Using the observation (i) we have already a natural fixed point condition for all angles. From the same argument, we see, however, that it is impossible to have a fixed point for the radial degree of freedom. To overcome this problem we introduce the hyperradius r, scaled by the excess energy ε,

$$r = R\varepsilon. \tag{3}$$

For any finite R the scaled hyperradius will approach $r^* = 0$ in the threshold limit $\varepsilon \to 0$.

The scaling Eq. (3) is not only necessary to obtain a radial fixed point condition. It also enables us to formulate the threshold fragmentation cross section in terms of the stability properties of the fixed point. They are determined by the behavior of the system under small perturbations $\delta\boldsymbol{y} = \boldsymbol{y} - \boldsymbol{y}^*$ about the fixed point, where $\boldsymbol{y}$ denotes the vector of all phase space variables, i.e., momenta and coordinates. Choosing appropriate values for δr we can show that the stability properties describe the fragmentation dynamics in the limit $\varepsilon \to 0$.

As already mentioned implicitly in (i), a fully fragmented state is enforced by $R \to \infty$ and all angles $\omega_i = \omega_i^*$ being fixed. In scaled coordinates an arbitrarily small but finite deviation $\delta r = \Delta$ from the fixed point is sufficient to have $\delta R = \delta r/\varepsilon \to \infty$ for $\varepsilon \to 0$.

For a fragmentation process to be possible, at least some of the constituents must be bound before the fragmentation. To break these bonds energy must be transferred to the bound subsystem. This requires an interaction, i.e., a nonvanishing potential $V \neq 0$ and in turn a finite hyperradius R_0. However, in scaled coordinates we have $\delta r^- = R_0\varepsilon$ which tends to zero for $\varepsilon \to 0$. Hence, the necessity of energy exchange for a fragmentation process to happen forces the system to the fixed point $r^* = 0$ for $\varepsilon \to 0$. For fragmentation induced by particle impact, the initial hyperradius R_0 must be chosen so large that the observables of the scattering process do not change with a small change of R_0. In practice, this means to calculate the cross section as a function of R_0 and to take the limit $R_0 \to \infty$ afterwards.

To summarize, in the threshold limit the time evolution of a small deviation δr from the fixed point is sufficient to describe the dynamical evolution of the system with initial and final deviations specified as

$$\delta r^- = R_0\varepsilon, \quad \delta r^+ = \Delta, \tag{4}$$

respectively, where Δ is an arbitrary constant. Working in energy-scaled coordinates and using a change of the total energy ε to describe the dynamical evolution of the system at a fixed energy as implied by Eq. (4) requires a careful separation of energy-dependent and energy-independent variables. Otherwise one leaves the energy shell ε for small deviations $\delta\boldsymbol{y}$ from the fixed point. In order to make sure that all variables except r are influenced by the change in ε we must construct a Hamiltonian $\mathscr{H}$ such that δr is orthogonal to all other deviations $\delta\boldsymbol{y}$. The Hamiltonian which fulfills the above requirements reads

$$h = (E - K)/V - 1 = 0 \tag{5}$$

which is essentially $-(H - E)/V$. We define new momenta as

$$p = P(-V)^{-1/2}, \tag{6}$$

$$p_i = P_{\omega_i}(-Vr^2)^{-1/2}, \tag{7}$$

where the reason for including an additional factor r^2 in p_i will become clear later. With these momenta the critical Hamiltonian reads

$$\mathscr{H} = p^2/2 + D(\boldsymbol{\omega}) + E_0\varepsilon/V(r/\varepsilon, \boldsymbol{\omega}) - 1 \equiv 0, \tag{8}$$

where

$$D = \sum_{i,j} l_{ij}(\boldsymbol{\omega}) p_i p_j \tag{9}$$

is the kinetic energy from the angular degrees of freedom and only the term $E/V = E_0\varepsilon/V$ carries an explicit dependence on ε. The usual centrifugal radial dependence $1/r^2$ has been absorbed in the p_i defined by Eq. (7) to avoid singularities when the fixed point $r^* = 0$ is approached.

3. Derivation of the threshold law

The equations of motion for the new phase space variables are most easily derived by determining first the corresponding equations for the old (canonical) phase space variables from the Hamiltonian h of Eq. (5) in the canonical way. In a second step, one takes the time derivative of the new momenta in terms of the old ones according to Eq. (2). Finally, we need a set of equations which behave regularly at the fixed point $r^* = 0$. This can be achieved by changing the time variable from the old time t defined by the Hamiltonian h of Eq. (5) to a new time defined by

$$\mathrm{d}t = \mathrm{d}\tau(-Vr^2)^{1/2}. \tag{10}$$

Denoting $\mathrm{d}x/\mathrm{d}\tau$ by $\dot{x}$, the equations of motion read finally

$$\dot{r} = pr,$$

$$\dot{p} = k(p^2/2 - 1) + D - p\Omega/2,$$

$$\dot{\omega}_i = \frac{\partial D}{\partial p_i},$$

$$\dot{p}_i = \frac{1}{V}\frac{\partial V}{\partial \omega_i} - \frac{\partial D}{\partial \omega_i} - p_i p(1 - k/2) - p_i\Omega/2, \tag{11}$$

where k and Ω are defined by

$$k = -\frac{R}{V}\frac{\partial V}{\partial R}; \quad \Omega = \sum_j \frac{\partial V}{\partial \omega_j}\frac{\dot{\omega}_j}{V}, \tag{12}$$

respectively. At threshold $\varepsilon = 0$ this dynamical system of the abstract form $\dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y})$ has a fixed point $\dot{\boldsymbol{y}} = 0$

with values

$$\omega_i^* \quad \text{from } \partial V/\partial\omega_i = 0,$$

$$p_i^* = r^* = 0,$$

$$p^* = \sqrt{2}, \tag{13}$$

where the value for p^* follows from the energy conservation according to the Hamiltonian Eq. (8).

Due to the scaling $r = R\varepsilon$ the fixed point $r^* = 0$ is realized for arbitrary unscaled R_0 at threshold $\varepsilon = 0$. For $R_0 \to \infty$ the fixed point at $\varepsilon = 0$ characterizes the fully fragmented system where all particles have infinite (unscaled) distance from each other, since all angles are fixed at ω_i^*. For small excess energies ε, deviations from the fixed point correspond to trajectories which do not contribute to the fragmentation since they fail to reach the fixed point for large interparticle distances due to its instability. The instability of the fixed point can be quantified by linearizing the equations of motion about the fixed point according to

$$\delta\dot{y}_i = \sum_j M_{ij}^* \delta y_j, \tag{14}$$

where $M_{ij}^* = \partial f_i/\partial y_j$ is the stability matrix at the fixed point. Diagonalizing the matrix $\mathscr{M}^*$ with components M_{ij}^* yields the Liapunov exponents λ_i as eigenvalues and the normal modes $\delta\gamma$ as linear combinations of the δy with the explicit time dependence

$$\delta\gamma_i(\tau) = \mathrm{e}^{\lambda_i\tau}\delta\gamma_i(0). \tag{15}$$

The Hamiltonian $\mathscr{H}$ has been constructed such that all normal modes with eigenvalues λ_i decouple from the radial normal mode whose differential equation $\delta\dot{r} = p^*\delta r$ follows directly from Eq. (11). Integration yields

$$\delta r(\tau) = \mathrm{e}^{\lambda_r\tau}\,\delta r(0), \tag{16}$$

with the radial Liapunov exponent $\lambda_r = p^*$. From Eq. (4) we may identify δr^+ with $\delta r(\tau)$ and δr^- with $\delta r(0)$. Thereby, we obtain the desired relation between energy change in ε and time evolution in τ negotiated by the scaling and dynamical evolution properties of the hyperradius,

$$\mathrm{e}^{\lambda_r\tau} = \frac{\delta r(\tau)}{\delta r(0)} \equiv \frac{\delta r^+}{\delta r^-} \propto \varepsilon^{-1}. \tag{17}$$

This relation states how $\tau \to \infty$ is connected with the threshold limit $\varepsilon \to 0$ through the instability of the hyperradius δr.

To derive the explicit form of the threshold law, we must formulate the classical fragmentation cross section for N particles. The threshold law is essentially independent of the actual initial state as expected for a critical phenomenon. Hence, at $\tau = 0$ with a finite hyperradius $R(\tau = 0) = R_0$ before the fragmentation has happened, we consider a general initial phase space distribution $\rho(R_0, \boldsymbol{y})$ parameterized by $6N - 8$ phase space variables $\boldsymbol{y}$. The y_j depend on the physical initial state and the process which induces fragmentation (e.g. particle impact or photon impact, etc.). At fixed hyperradius R_0 the y_i completely determine the distribution ρ since the radial momentum P is used to satisfy energy conservation. The total fragmentation cross section is the integral over the differential cross section at all final hyperangles $\boldsymbol{\alpha}^+$ and can be expressed as

$$\sigma \propto \lim_{\tau\to\infty} \int \mathrm{d}\boldsymbol{\alpha}^+ \int \mathrm{d}\boldsymbol{y}\, \rho(R_0, \boldsymbol{y})\delta(\boldsymbol{\alpha}(\tau) - \boldsymbol{\alpha}^+). \tag{18}$$

The δ-functions contain the entire, complicated dynamics of the N-particle system. To resolve it we choose $N - 2$ (the dimension of $\boldsymbol{\alpha}$) of the integration variables $y_j(0)$ for which the Jacobi matrix $J_{ij} = \partial\alpha_i(\tau)/\partial y_j(0)$ has nonvanishing determinant $|J_{ij}| \neq 0$. With the $\alpha_i(\tau)$ as new integration variables all integrations can be performed and we are left with

$$\sigma \propto \int \mathrm{d}\boldsymbol{\alpha}^+\, |J_{ij}|^{-1}_{\boldsymbol{\alpha}(\tau)=\boldsymbol{\alpha}^+}. \tag{19}$$

From Eq. (17) we know that in the threshold limit it is sufficient to evaluate the dynamical Jacobi matrix in linear approximation which reads with the help of Eq. (15)

$$J_{ij} = \sum_k A_{ik}\mathrm{e}^{\lambda_k\tau}B_{kj} \tag{20}$$

where the matrices $A_{ij} = \partial\alpha_i/\partial\gamma_j$ and $B_{ij} = \partial\gamma_i/\partial y_j$ represent coordinate transformations at times (τ) and (0), respectively. For large times τ, the largest Liapunov exponents will dominate. Assuming an ordering $\lambda_1 > \lambda_2, \ldots$, we obtain for the determinant

$$|J_{ij}(\tau \to \infty)| \propto \exp\left[\sum_i^{N-2} \lambda_i\tau\right]. \tag{21}$$

The $N - 2$ terms in the sum reflect the fact that the determinant measures an $N - 2$ dimensional volume which is spanned by the $N - 2$ normal modes with the largest Liapunov exponents. Inserting Eq. (21) into

Eq. (19) and using the relation Eq. (17) between τ and ε we arrive at

$$\sigma(\varepsilon \to 0) \propto \varepsilon^{\Lambda/\lambda_r}, \quad (22)$$

where

$$\Lambda = \lim_{R_0 \to \infty} \sum_i^{N-2} \lambda_i(R_0). \quad (23)$$

The threshold law has a simple interpretation: The amount of fragmentation is decided by the competition of two processes, firstly, the mutual separation of the particles with a growing hyperradius R, and secondly, the departure from the fixed point of fragmentation in angular directions. Threshold fragmentation is, relatively speaking, a probable event, if the rate of radial growth λ_r of the system is large compared to Λ, the rate with which the dynamical system leaves the fixed point. Finally, with an increasing number of particles N, there are more possibilities for partial fragmentation which lower the cross section for complete fragmentation by more unstable directions contributing to the sum in Eq. (23).

In the following section we will illustrate the result obtained in a rather abstract way for three charged particles.

4. General form of the threshold law for three-particle systems of two degrees of freedom

To arrive at a compact form and an analytically solvable fixed point condition we restrict ourselves to A–B–A systems. Assume a collinear arrangement of the three particles where B with mass m_B is located in between the two particles of sort A with mass m_A described by the two distances r_{AB} and r_{BA}. In mass-weighted hyperspherical coordinates (r,α) this geometrical arrangement can be parameterized in a symmetric and compact way by

$$r_{AB} = r\sin(\alpha_0 - \alpha),$$

$$r_{BA} = r\sin(\alpha_0 + \alpha),$$

$$\alpha_0 = \frac{1}{2}\arctan\sqrt{\frac{m_B}{m_A}\frac{2m_A + m_B}{m_A}}, \quad (24)$$

where $\alpha \in [-\alpha_0, \alpha_0]$. The Hamiltonian Eq. (5) formulated with momenta P and P_α reads

$$h_2 = -(P^2/2 + P_\alpha^2/(2r^2) - E)/V - 1 \equiv 0 \quad (25)$$

where $V = V(r/\varepsilon, \alpha)$ is the total potential in scaled coordinates. The equations of motion are now,

$$\frac{\mathrm{d}P}{\mathrm{d}t} \equiv -\frac{\partial h_2}{\partial r} = \varepsilon^{-1}\frac{V_r}{V} - \frac{P_\alpha^2}{Vr^3},$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} \equiv \frac{\partial h_2}{\partial P} = -\frac{p}{V},$$

$$\frac{\mathrm{d}P_\alpha}{\mathrm{d}t} \equiv -\frac{\partial h_2}{\partial \alpha} = \frac{V_\alpha}{V},$$

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} \equiv \frac{\partial h_2}{\partial P_\alpha} = -\frac{P_\alpha}{Vr^2}, \quad (26)$$

where V_x denotes $\partial V/\partial x$. Calculating the time derivative of the new momenta Eq. (2) with the help of Eq. (26) and formulating the equations of motion in the new time variable according to Eq. (10) yields the coupled first-order differential equations which are not singular at the fixed point (we define $p_1 \equiv p_\alpha(-Vr^2)^{-1}$):

$$\dot{r} = pr,$$

$$\dot{p} = k(p^2/2 - 1) + p_1^2 - \frac{p_1 p}{2}\frac{V_\alpha}{V},$$

$$\dot{\alpha} = p_1,$$

$$\dot{p}_1 = \frac{V_\alpha}{V}(1 - p^2/2) - p_1 p(1 - k/2). \quad (27)$$

The potential energy of the collinear Hamiltonian for two electrons (particles A, mass $m_A = 1$) and an infinitely heavy nucleus B of charge Z is given by

$$v(r_{AB}, r_{BA}) = -Z/r_{AB} - Z/r_{BA} + |r_{AB} + r_{BA}|^{-1}. \quad (28)$$

It is easy to see that $V_\alpha = 0$ is fulfilled for $\alpha^* = 0$. Hence, in the coordinates chosen, the relevant fixed point for fragmentation is given by $p_1^* = r^* = \alpha^* = 0$ and $p^* = \sqrt{2}$ enforced by energy conservation. The stability of the fixed point is calculated from small deviations

$$\delta\dot{p} = 0, \quad (29)$$

$$\delta\dot{r} = p^*\,\delta r, \quad (30)$$

$$\delta\dot{p}_1 = A_{11}\,\delta p_1 + A_{12}\,\delta\alpha, \quad (31)$$

$$\delta\dot{\alpha} = A_{21}\,\delta p_1 + A_{22}\,\delta\alpha \quad (32)$$

with the elements

$$A_{11} = p^*/2, \quad (33)$$

$$A_{12} = \frac{\partial^2 V}{\partial \alpha^2}\frac{1}{V} \equiv \frac{V^*_{\alpha\alpha}}{V^*}, \tag{34}$$

$$A_{21} = 1, \tag{35}$$

$$A_{22} = 0 \tag{36}$$

of the matrix $\mathscr{A}$. Eq. (4) shows the decoupling of δr from the other degrees of freedom which has been achieved by construction of the critical Hamiltonian. The decoupling leads in connection with $\delta\dot{p} = 0$ immediately to the integrated time evolution of Eq. (16) with $\lambda_r = p^* = \sqrt{2}$. The time evolution of the hyperangle α is obtained by calculating the determinant $\det(\mathscr{A} - \lambda)$ which leads to a quadratic secular equation for λ with the solutions

$$\lambda_\pm = -\frac{\lambda_r}{2}\left(1 \pm \sqrt{1 + \frac{16V_{\alpha\alpha}}{V[p^*]^2}}\right), \tag{37}$$

where use has been made of the fact that $p^* = \lambda_r$. Since we have only one angular degree of freedom the sum for the rate in Eq. (23) contains only one term and

$$\Lambda = \lim_{R\to\infty} \lambda_-(R). \tag{38}$$

With Eqs. (37) and (38) we can directly evaluate the threshold exponent for specific interactions. For the interaction Eq. (28) the threshold law reads $\sigma \propto \varepsilon^\beta$ with

$$\beta = \frac{\lambda_-}{\lambda_r} = \frac{1}{4} - \frac{1}{4}\sqrt{\frac{100Z-9}{4Z-1}}, \tag{39}$$

which is exactly Wannier's result [1].

5. Universal shape function for ionization cross sections

The experimental verification of the threshold cross section $\sigma(\varepsilon) \propto \varepsilon^\beta$ is due to its very nature difficult because the signal is small, even zero at threshold $\varepsilon = 0$. However, the threshold behavior influences significantly the shape of the fragmentation cross section over a wide energy range. This influence can be used to verify the threshold exponents indirectly. Moreover, the influence manifests itself in a universal shape of the ionization cross section which only depends on the threshold exponent β [4],

$$\sigma(E) = \sigma_M f(E/E_M), \tag{40}$$

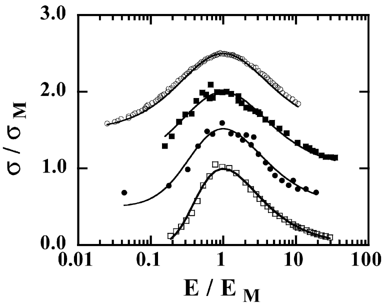

Fig. 1. Double-ionization cross sections for H^- (circles) [6], O^- (squares), and C^- (triangles) [7] in terms of the maximum values E_M, σ_M. In addition the shape function Eq. (41) is shown with a solid line.

where the shape function is given by

$$f(x) = x^\beta \left(\frac{\beta+1}{\beta x+1}\right)^{\beta+1}. \tag{41}$$

The shape is obtained by combining the low-energy power-law behavior and the classical high-energy behavior $\sigma(\varepsilon \gg 1) \propto 1/\varepsilon$ as it has been derived by Thomson in 1912 [5]. The shape function is *parameter free* if expressed in dimensionless variables, i.e., the cross section is expressed as $y = \sigma/\sigma_M$ and the energy is expressed as $x = E/E_M$, where E_M is the energy where the cross section has its maximum, and σ_M is this maximum value. Fig. 1 shows examples for different three-body fragmentations. However, the shape function is not restricted to three charged fragments. Since the high-energy behavior is determined by the projectile the shape function remains the same even for more fragments, only the exponent β changes. Fig. 2 presents data for double-electron detachment from negative ions by electron impact which leaves four charged fragments [12].

6. Summary

We have described threshold fragmentation into three or more charged particles as a critical phenomenon which motivates that the fragmentation cross section has the closed form of a power law as a function of excess energy. Moreover, we have derived the critical exponent as the ratio of two rate constants which characterize the stability of the fixed point responsible for the criticality of threshold

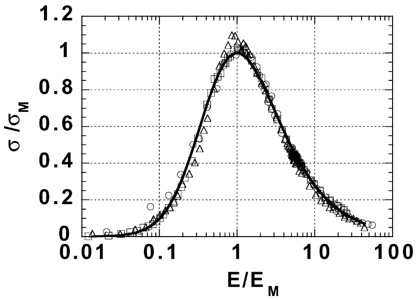

Fig. 2. Experimental cross sections for the ionization of hydrogen plotted in scaled coordinates $y = \sigma/\sigma_M$ versus E/E_M. The solid line is the shape function Eq. (41). The projectiles are protons [8] (open squares), positrons ($y + 0.5$, [9], filled circles), antiprotons with helium as a target ($y + 1$, [10], filled squares), and electron impact ($y + 1.5$, [11], open circles).

fragmentation. This ratio shows that threshold fragmentation can be viewed as the competition between the radial expansion of the system described by one rate and the rate for failing to reach the fixed point of fragmentation during the radial expansion. This second rate is expressed as a sum over Liapunov exponents at the fixed point. While this result, obtained for systems of only a few degrees of freedom, is theoretically intriguing, its value for experiments lies in the influence of the threshold behavior on the form of the ionization cross section over a large energy range. This influence has been demonstrated by formulating a semiempirical ionization shape function, only dependent on the threshold exponent β. Although so far this shape lacks a rigorous theoretical derivation it has been very successful for describing and interpreting experimental cross sections.

7. Ucited reference

[13]

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