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On the multifractal nature of fully developed turbulence and chaotic systems

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Abstract. It is generally argued that the energy dissipation of three-dimensional turbulent flow is concentrated on a set with non-integer Hausdorff dimension. Recently, in order to explain experimental data, it has been proposed that this set does not possess a global dilatation invariance: it can be considered to be a multifractal set. In this paper we review the concept of multifractal sets in both turbulent flows and dynamical systems using a generalisation of the β-model.

1. Introduction

One of the most tested hypotheses in the theory of fully developed turbulence is that the small-scale statistics of turbulent flows obeys universal scaling properties. This is the celebrated Kolmogorov theory (1941, hereafter K41) which is still the only prediction made on the statistical properties of turbulence. The K41 theory is based upon the concept of self-similarity of the inertial range and upon the dependence of the probability distribution on the energy dissipation. Deviations of the K41 theory are commonly argued to be given by intermittency in the flow as first pointed out by Landau (Landau and Lifshitz 1971). Since Landau's remark there has been a great effort in generalising the K41 theory to include the intermittency correction, however no definite theoretical framework has been found to deal with the problem of intermittency. It has been argued that intermittency is due to the singularity of the Navier–Stokes equations in the small viscosity limit. It has been proposed by Mandelbrot (1974) that singularities are concentrated on a set $A \subset R^3$ with non-integer Hausdorff dimension. In § 2 we shall see how this is related to the scale invariance of the Navier–Stokes equations.

It is important to understand how the dynamical properties of nonlinear energy transfer among the various scales of motions determine the geometrical properties of the set A. Frisch *et al* (1978, hereafter FNS) have clarified this point introducing the well known β-model, also reviewed in § 2. The fundamental ingredients of these models is that there is a detailed balance of energy transfer in the inertial range. For detailed balance of energy transfer we mean that there is an exact balance in any shell k, $k+\mathrm{d}k$ ($k =$ wavenumber) between input/output energy. If we relax this constraint, we must then introduce a hierarchy of sets A_i on which singularities with different scaling properties are concentrated. The mechanism of energy transfer has to be constrained, not in detailed balance between local interaction in the k-space, but on the average. In other words we look at the history of the nonlinear interactions both in space and in time. It turns out that to this aim we need to introduce the concept of multifractal sets as defined in Frisch and Parisi (1983) and reviewed in § 2. Whether turbulence can be described by ordinary fractal sets or multifractal sets can be inferred only from experimental data at this stage. Within the present experimental results we show that there are some indications that multifractal sets are indeed necessary to describe the properties of structure functions in turbulent flows. In § 3 we propose a simple model, the random β-model, which generalises the results of FNS to take into account the multifractal structure of turbulence.

Multifractal sets, which originally were embedded in the weight curdling of Mandelbrot (1974), have been recently invoked by Paladin and Vulpiani (1984) to study the attractor sets in deterministic chaotic systems. It is the aim of this paper to review the concept of multifractal sets not only for what concerns the turbulence theory but also in connection with strange attractors of dynamical systems. This is done in § 4 together with some numerical analysis of simple attractors.

2. Turbulence and multifractal sets

Let us consider the Navier–Stokes equations:

$$\partial_t V + (V\cdot\nabla)V = -\nabla p/\rho + \nu\,\Delta V. \tag{2.1}$$

In the limit $\nu \to 0$ the equations (2.1) are formally invariant under the scaling transformations (see Frisch 1983):

$$r \to \lambda r, \qquad V \to \lambda^h V \qquad \lambda > 0, \qquad t \to \lambda^{1-h} t. \tag{2.2}$$

For finite ν we can still ensure invariance of equations (2.1) if

$$\nu \to \lambda^{h+1}\nu. \tag{2.3}$$

Note that the Reynolds number VL/ν is invariant under transformations (2.2) and (2.3). Assuming that small-scale turbulence is statistically invariant under the above scaling law, one can select h by using physical arguments. Kolmogorov (1941) proposed that the scaling laws of turbulence preserve energy transfer, assuming that nonlinear interactions are local in the k-space. This assumption implies that energy dissipation ε is invariant under the scaling laws (2.3). By definition $\varepsilon = \nu\langle(\nabla V)^2\rangle$ where $\langle\ldots\rangle$ denotes ensemble average. It follows that:

$$\varepsilon \to \lambda^{3h-1}\varepsilon. \tag{2.4}$$

The invariance of ε implies $h = 1/3$. The K41 theory has strong implications for the velocity gradients ∇V. Let us consider the quantity

$$\Delta V/\Delta x^{1/3} = [V(x) - V(y)]/(x-y)^{1/3}.$$

Scaling laws (2.3) with $h = 1/3$ imply

$$\lim_{x\to y} \Delta V/\Delta x^{1/3} \neq 0. \tag{2.5}$$

Thus the velocity gradient is singular. However the above considerations do not imply that the set of singular points of the Navier–Stokes equations are space-filling as originally assumed by Kolmogorov. Following Mandelbrot (1974), we can define the Hausdorff dimension D of the set of singular points (2.5). If $D<3$ then the probability for a point to be singular behaves as λ^{3-D}. It follows that energy dissipation is a fluctuating quantity in space. Its probability distribution determines the value of D. This is the physical basis of Landau's remark.

A clarifying picture of the mechanism underlying the above ideas is obtained following FNS. Let us consider the scales:

$$l_n = l_0 2^{-n}$$

where l_0 is the scale on which energy is injected. Nonlinear interactions produce on the average N eddies of scale l_{n+1} i.e. energy is transferred from scale l_n to the N eddies of scale l_{n+1}. These N eddies occupy a fraction β $(0\leqslant\beta\leqslant 1)$ of the l_n-eddy volume. After n generations the volume occupied by 'active' eddies is β^n. If V_n is the velocity difference of an active l_n-eddy, then the energy per unit mass on scale l_n is

$$E_n \sim \beta^n V_n^2,$$

and the energy transfer is

$$\varepsilon_n \sim \beta^n V_n^3/l_n.$$

Assuming that

$$\varepsilon_n \sim \varepsilon = \text{constant}, \tag{2.6}$$

we obtain $V_n \sim \varepsilon^{1/3} l_n^{1/3} P^{-1/3}$ where $P\sim(l_n/l_0)^{3-D}$ and $\beta = 2^{D-3}$. Benzi and Vulpiani (1980) have estimated $D \simeq 3-2/3$ in good agreement with known data on the correlation functions of ε. Either the β-model and the scaling law (2.3) with $h=1/3$ predicts a linear behaviour of the coefficients

$$\langle\Delta V^p\rangle \sim r^{\zeta_p}, \qquad \zeta_p = hp+3-D_{\mathrm{F}}, \qquad h=(D_{\mathrm{F}}-2)/3.$$

In figure 1 we report data on ζ_p for various experimental tests (Anselmet *et al* 1983). Although a linear fit is not inconsistent with the experimental accuracy, there is a

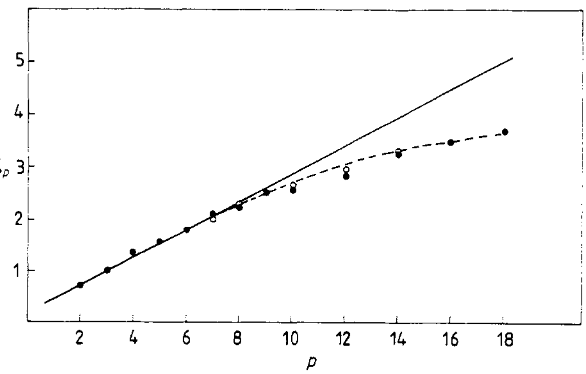

Figure 1. ζ_p against p. Dots and circles represent experimental data by Anselmet *et al* (1983). Full line is the β-model of FNS with $D_{\mathrm{F}}=2.83$. The broken line refers to equations (3.8) and (3.9) with $x=0.125$.

tendency for ζ_p to behave in a nonlinear way. If this is assumed to be the case, it follows that neither equation (2.4) nor equation (2.6) is valid. In other words the rate of energy transfer is not constant among the various scales of motion but fluctuates both in space and in time. This idea has been originally proposed by Mandelbrot (1974). A simple way to include fluctuations in the energy dissipation within the scaling laws (2.2), (2.3) has been recently proposed by Frisch and Parisi (1983, hereafter FP). Let us define $S(h)$ the set of points for which

$$\lim_{x\to y} [V(x)-V(y)]/(x-y)^h \neq 0$$

and let us denote by $d(h)$ the Hausdorff dimension of $S(h)$. The Kolmogorov theory simply implies that $d(h)=3\theta(h-1/3)$ while the β-model gives $d(h)= D\theta[h-(D-2)/3]$. Existence of singularities with arbitrary exponents h is consistent with the Navier–Stokes equations in the limit $\nu\to 0$. Thus the possibility that $d(h)$ is not a step function of h could be embedded into the scaling law of the velocity field. Generalising the considerations done at the beginning of this section, we can assume that the probability for a point to belong to $S(h)$ is proportional to $\lambda^{[3-d(h)]}$. It follows that

$$\langle \Delta V^p(r)\rangle \propto \int \mathrm{d}\mu(h) r^{|ph+3-d(h)|} \tag{2.7}$$

where $\mu(h)$ is a measure concentrated on the region where $d(h)>0$. The RHS of equation (2.7) can be estimated using a saddle-point technique. We obtain

$$\zeta_p = \min_h[ph+3-d(h)]. \tag{2.8}$$

For a proper choice of $d(h)$, equation (2.8) might fit the experimental data of figure 1. Physically, equation (2.8) implies that for a given value of p, ζ_p is dependent on a particular value of h, i.e. a particular value of $S(h)$. Hence the kind of instabilities needed to set up the sets $S(h)$ are picked out by the moments of the velocity differences. We mean, for example, instabilities which lead to vortex sheets or vortex tubes (for a review see Monin and Yaglom (1975)). Figure 1 can then be interpreted as the evidence of different mechanisms acting on the flow to select the probability distribution of energy transfer and dissipation. It is therefore clear that the β-model of FNS is not able to take into account the different nature of energy transfer. In the next section we discuss how to improve the β-model to deal with different sets of singularities. The idea that different moments of strongly fluctuating distribution can be dominated by different singularities has been explored, in a different context, by Berry (1977 and 1982).

3. A random β-model

Let us assume that, in the spirit of Mandelbrot's weight curdling, the contraction factors β are independent random variables, which can take different values in each eddy i at the step n of the energy cascade. The $\beta_n(i)$'s are distributed according to a given probability distribution $P(\beta)$. In this way the geometrical structure of intermittency does not possess a global dilatation invariance. The number of active eddies which are generated in a step is not fixed by a parameter as for ordinary 'homogeneous' fractals. The rules which generate multifractal inhomogeneous sets are drawn at random at each step in length scale. Figure 2 shows two different naive pictures of intermittency according to either the deterministic or the random β-model. It is interesting to compute the fractal dimension of the multifractal set which is defined by

$$\langle N_n \rangle \sim l_n^{-D_F} \tag{3.1}$$

where N_n is the number of active eddies at the nth step and $\langle\ \rangle$ indicates a space average, see Mandelbrot (1982, p 211 where D_F is called the similarity dimension). We shall also use the average $\{\ \}$ on the distribution $P(\beta)$

$$\{f\} \equiv \int \mathrm{d}\beta\, P(\beta) f(\beta).$$

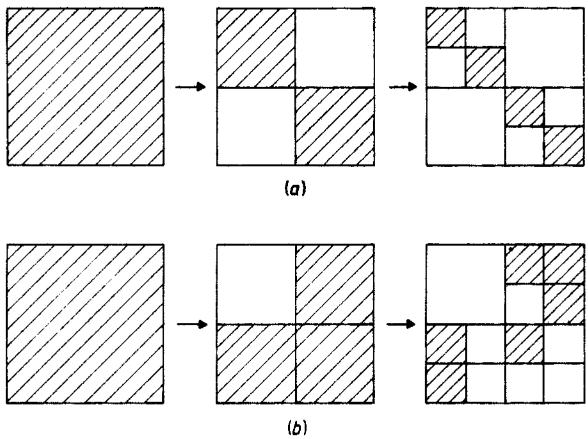

Figure 2. (a) Schematic view of the β-model and (b) compared with the random β-model. The shaded areas are the zones active during the fragmentation process.

An eddy of size $l_{j+1} = l_j/2$ covers a fraction $B_j = \Sigma_i \beta_j(i)/N_j$ of the volume occupied by its hypercube 'father'. It follows that the number of active eddies after n steps is given by

$$N_n = 2^{3n} \prod_{j=1,n} B_j. \tag{3.2}$$

We can easily average (3.2), noting that the β's are independent random variables:

$$\langle N_n \rangle = 2^{3n} \left\langle \prod_{j=1,n} B_j \right\rangle \sim l_n^{-(3+\log_2\{\beta\})}.$$

We have transformed the space average in β-average. By definition (3.1) we can obtain

$$D_F = 3 + \log_2\{\beta\}. \tag{3.4}$$

D_F cannot completely characterise an 'inhomogeneous' fractal because it does not give complete information on the probability distribution of β. The main point of this section is to show that this information is provided by the exponents ζ_p.

Let us denote by $l_n(k)$, $k = 1, \ldots, N_n$, the N_n active eddies at the nth step. Each $l_n(k)$ generates eddies of size $l_{n+1}(k)$ where k indicates its origin. The rate of energy transfer is constant among $l_n(k)$ and $l_{n+1}(k)$:

$$V_n^3(k)/l_n(k) = \beta_{n+1}(k)\, V_{n+1}^3(k)/l_{n+1}(k). \tag{3.5}$$

This relation implies that the velocity difference $\delta V(l_n) \equiv V_n$ in an eddy generated by a particular set of fragmentations $\beta_1, \beta_2, \ldots, \beta_n$, is

$$V_n \sim l_n^{-1/3}\left(\prod_{i=1,n} \beta_i\right)^{-1/3}. \tag{3.6}$$

The structure functions are then

$$\langle|\delta V(l_n)|^p\rangle = \int \prod_{i=1,n} \mathrm{d}\beta_i P(\beta_1, \ldots, \beta_n)\beta_i |V_n|^p. \tag{3.7}$$

Because we assumed that there are no correlations among different steps of the fragmentation process, it follows that

$$\prod_{i=1,n} \mathrm{d}\beta_i\, P(\beta_1, \ldots, \beta_n) = \prod_{i=1,n} P(\beta_i)\, \mathrm{d}\beta_i.$$

From (3.7) we can compute the exponents ζ_p:

$$\zeta_p = p/3 - \log_2\{\beta^{(1-p/3)}\}. \tag{3.8}$$

The probability distribution $P(\beta)$ is known in principle from the knowledge of all the β moments, i.e. of all the exponents ζ_p. Figure 1 shows that the simple form

$$P(\beta) = x\delta(\beta - 0.5) + (1-x)\delta(\beta - 1) \tag{3.9}$$

leads (with $x = 0.125$) to a good fit to the available experimental data, x being the only free parameter. There is no good reason to choose a two-step probability distribution for β, of course. We have assumed that an active eddy can generate either velocity sheets ($\beta = 0.5$) or space-filling Kolmogorov-like eddies ($\beta = 1$) (see Saffman 1968). We see, by comparing relation (3.8) and (3.4), that in our model the fractal dimension is

$$D_{\mathrm{F}} = 3 - \zeta_0, \tag{3.10}$$

nevertheless D_{F} is often computed by the energy dissipation correlation

$$\langle\varepsilon(x+r)\varepsilon(x)\rangle \sim r^{-\mu}. \tag{3.11}$$

$D^* \equiv 3 - \mu$ is considered equal to D_{F}. FNS have shown that under general conditions

$$D^* = 1 + \zeta_6. \tag{3.12}$$

This relation is still valid in our model, with the further inequality

$$D_{\mathrm{F}} = 3 + \log_2\{\beta\} \geqslant D^* = 3 - \log_2\{\beta^{-1}\}. \tag{3.13}$$

The equality $D_{\mathrm{F}} = D^*$ holds only in the deterministic β-model. From the fit of the data of the structure functions given in figure 1, we have obtained

$$D_{\mathrm{F}} = 2.91, \qquad D^* = 2.83. \tag{3.14}$$

This result is an indirect check of the multifractal nature of fully developed turbulence.

4. Fractal structure of strange attractors

The scenario of the random β-model is quite general and can be extended to the analysis of dynamical systems. Indeed the chaotic motions often lie on complicated manifolds of the phase space, called strange attractors, which can have an intricate multifractal structure. The fractal dimension D_F cannot fully characterise an attractor, and further the FNS β-model does not describe the intermittency in a satisfactory way. We shall therefore introduce a set of easily computable exponents generalising the fractal dimension which are defined in terms of a 'local density' $n(r)$.

Let us consider a time series of points $X_i \equiv X(i\Delta t)$, $(i = 1, 2, \dots N)$ of the dynamical system

$$dX/dt = f(X), \qquad \text{where } X \text{ belongs to } R^d. \tag{4.1}$$

The fraction of points which are contained in a hypersphere of radius r and centre X_i is

$$n_i(r) = \sum_{j\neq i} \theta(r - |X_i - X_j|)/(N-1). \tag{4.2}$$

The moments of such a local density are (via a space average)

$$\langle n(r)^q \rangle = \lim_{N\to\infty} \sum_{i=1,N} n_i(r)^q/N. \tag{4.3}$$

We define a new set of exponents $\phi(q)$ by the relation:

$$\lim_{r\to 0} \langle n(r)^q \rangle \sim r^{\phi(q)}. \tag{4.4}$$

In a homogeneous fractal, (see figure 2(*a*)), $n(\lambda r)$ has the same statistical properties as $n(r)\lambda^{D_F}$ and this implies:

$$\phi(q) = D_F q. \tag{4.5}$$

On the contrary, attractors do not possess a global dilatation invariance and it is only possible to show that $\phi(q)$ is convex in q by a general theorem of probability (Feller 1971). It is worth pointing out that $\phi(1)$ is the exponent ν proposed by Grassberger and Procaccia (1983) to estimate the fractal dimension of attractors. $\phi(q)$ is plotted versus q for the Lorenz model and the Hénon map in figure 3. One sees that $\phi(q)$ is nearly linear at $|q| \leq 1$ but deviations from the line $D_F q$ appear at larger values of $|q|$. It is evident that the $\phi(q)$ are analogous to the exponents ζ_p for the velocity fluctuations in a turbulent flow: the ζ_p are linear in p in the FNS scheme where the energy dissipation is concentrated in a homogeneous fractal set. Grassberger (1983) has recently introduced some exponents essentially equivalent to ours. We note that the $\phi(q)$ as defined in Paladin and Vulpiani (1984) are easier to compute than the Grassberger ones. Indeed we do not need to use the box-counting method which is not easily handled for topological dimension $d > 3$. We shall see that our approach shows a connection between the structure of the attractor and the dynamics of the system.

We have to relate the exponent ϕ to the dynamical properties of the system (4.1) by an adaptation of the model proposed in § 3. We shall assume therefore that the same statistics are obtained by considering the positions in the phase space, at large times, of N points which are uniformly distributed in a hypercube of size l_0 at the initial time, instead of the N position at times $i\Delta t$ of the evolution of one point. This

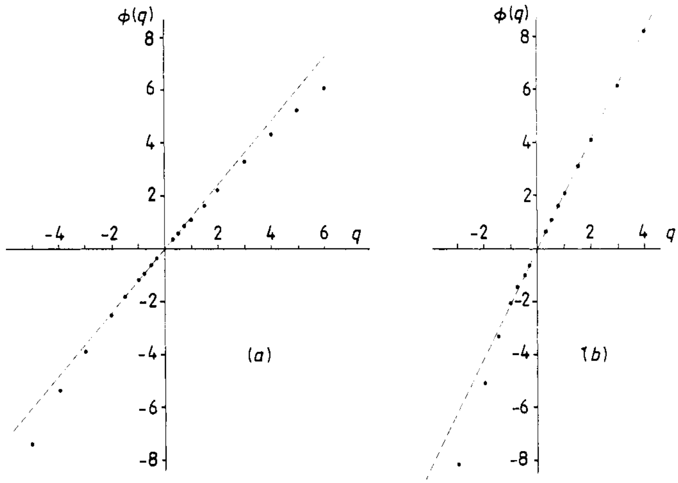

Figure 3. Variation of ϕ against q. Broken lines correspond to the line $\phi = qD_F$. Dots are $\phi(q)$ computed with $N = 10^4$ for: (a) Hénon map ($a = 1.2$, $b = 0.3$) (Hénon 1976); and (b) Lorenz model ($r = 28$) (Lorenz 1963).

possible because of the mixing property which is satisfied for chaotic systems. Let us describe the dynamics by a fragmentation mechanism which allows us to apply the formalism of the preceding section. At time Δt_1 after the initial time, our N points in the phase space of the system (4.1) will be distributed in $N_1 = a^d\beta_1$ hypercubes of size $l_1 = l_0/a$ ($a > 1$) where β_1 is a parameter given by the dynamics. Each new hypercube generates $N_2 = a^d\beta_2$ hypercubes of size $l_2 = l_1/a$ after a time Δt_2. The β_2's depend, as before, on their particular hypercube 'father'. We can iterate this process being careful to choose the breaking times Δt_i so that $l_{i+1} = l_i/a$. In this way the initial hypercube of size l_0 is reduced to a large number of hypercubes of size $l_n = a^{-n}l_0$ after n steps. Each hypercube has its own history determined by a succession $\beta_1(\alpha), \ldots, \beta_n(\alpha)$ where α indicates the history which is considered. We have now to impose conservation of the number of points at each step. We have

$$l_{j-1}^d\rho_{j-1}(\alpha) = [(l_{j-1}/l_j)^d\beta_j(\alpha)]l_j^d\rho_j(\alpha) \tag{4.6}$$

where $\rho_j(\alpha)$ is the point density of a hypercube of size l_j obtained by a fragmentation history α. It follows from (4.6) that

$$\rho_j(\alpha) \sim \left[\prod_{i=1} \beta_i(\alpha)\right]^{-1}. \tag{4.7}$$

We can estimate (4.4) putting $r = l_n = l_0 a^{-n}$:

$$\langle n(l_n)^q\rangle \sim \langle (l_n^d\rho_n)^q\rangle = l_n^{dq}\langle\rho_n^q\rangle \sim a^{-ndq}\left\langle\left[\prod_{i=1,n}\beta_i(\alpha)\right]^{-q}\right\rangle, \tag{4.8}$$

where the β_i's can be assumed to be independent random variables. We can therefore perform the average (4.8) as an average on the probability distribution for the β of a single fragmentation:

$$\langle n(r)^q\rangle \sim a^{-ndq}\langle\beta^{-q}\rangle^n \sim a^{-n[dq-\log_a\{\beta^{-q}\}]}. \tag{4.9}$$

The exponents ϕ are thus

$$\phi(q) = \mathrm{d}q - \log_a\{\beta^{-q}\}. \tag{4.10}$$

Relation (4.10) is quite analogous to the relation (3.8) in § 3. One has in the case of a homogeneous fractal:

$$\beta_i(\alpha) = a^{(D_\mathrm{F}-d)}, \qquad \text{for any } \alpha \text{ and } i \tag{4.11}$$

and (4.5) trivially follows from (4.10). We use now the definition (3.1) to compute the fractal dimension of an attractor in our model. The number $N(r)$ of hypercubes of size $r(=l_n)$ necessary to cover an attractor, in the limit $r \to 0(n \to \infty)$, is

$$N(l_n) \sim a^{nd} \prod_{i=1,n} B_i, \tag{4.12}$$

where B_i has been defined in § 3 as

$$B_i = \sum_{i=1,N_i} \beta_j(i)/N_j.$$

Averaging (4.12) one obtains

$$\langle N(l_n)\rangle \sim a^{n[d+\log_a\{\beta\}]}. \tag{4.13}$$

The relation

$$D_\mathrm{F} = -\phi(-1) \tag{4.14}$$

follows from the definition of the ϕ's in (4.10), the estimate (4.14) is in good agreement with the results obtained by the box-counting methods (Russell *et al* 1980):

$$\text{Hénon}(a=1.2,\ b=0.3) \qquad -\phi(-1) = 1.20 \pm 0.01,$$

$$\text{Lorenz}(r=28) \qquad -\phi(-1) = 2.06 \pm 0.01.$$

The inequality $\{\beta\}^{-1} \leqslant \{\beta^{-1}\}$ implies $\nu \leqslant D_\mathrm{F}$. ν gives therefore a higher weight to the lower density regions of an attractor. $\{\beta^{-1}\}$ can however be a more interesting parameter than $\{\beta\}$ as the average value and the most probable one can differ. It is amusing to note that the estimate of D_F by ν corresponds to the estimate of D_F by D^* in fully developed turbulence. Both $\phi(1)$ and ζ_6 are determined by $\{\beta^{-1}\}$ while D_F is related to $\{\beta\}$.

We want finally to discuss, in terms of our model, Mori's (1980) estimate of the fractal dimension from the Lyapunov exponents (say $\lambda_1 \geqslant \lambda_2 \ldots \geqslant \lambda_n$) of a dynamical system:

$$D_\mathrm{F} = d_+ + d_0 + d_-\left[\sum_{i=1,d_+} \lambda_i \bigg/ \left|\sum_{j=1+d_0+d_+,d} \lambda_j\right|\right], \tag{4.15}$$

where d_+, d_0 and d_- are the number of λ_i which are respectively greater than, equal to or less than zero.

Let us define A_{ij}, the matrix that describes the linearised evolution of the system around the time t_n:

$$A_{ij} = \partial f_i/\partial x_j|_{t=t_n},$$

where $t_n = \Sigma_{i=1\ldots n} \Delta t_i$. Let $K_n(i)$ be the eigenvalues of A_{ij}. It is possible to write B_n

as a function of $K_n(i)$:

$$B_n = \sum_{j=1,N(l_{n-1})} \exp\left(\sum_{i=1,d} K_n^j(i)\Delta t^j_n\right) \Big/ N(l_{n-1}) \tag{4.17}$$

$K_n^j(i)$ is the ith eigenvalue of the matrix A_{ij} computed in the centre of the jth hypercube ($i=1, d; j=1, N(l_n-1)$), Δt^j_n is the time needed to reduce the size of the jth hypercube from l_{n-1} to l_n i.e. the time needed to have the fragmentation. Δt^j_n is determined by the eigenvalues $K^j_n(i)<0$:

$$\Delta t^j_n = \ln a/|\kappa_n(j)| \tag{4.18}$$

where $\kappa_n(j)$ is the local contraction rate at the time t_n around the centre of the jth hypercube.

An estimate of $\kappa_n(j)$ is not trivial. Mori assumed for $\kappa_n(j)$ an average over the negative eigenvalues of the matrix A_{ij}:

$$\kappa_n(j) = \sum_{i=1+d_0+d_+,d} |K^j_n(i)|/d_-. \tag{4.19}$$

By equations (4.12), (4.17) and (4.18) we have

$$N(l_n)/N(l_{n-1}) = \sum_{j=1,N(l_{n-1})} (l_{n-1}/l_n)^{[d+\Sigma_{i=1,d}(K^j_n(i)/\kappa_n(j))]}/N(l_{n-1}). \tag{4.20}$$

As $\langle N(l_n)\rangle \sim l_n^{-D_F}$ the fractal dimension D_F becomes

$$D_F = d + \ln_a\langle a^{[\Sigma_{i=1,d} K^j_n(i)/\kappa_n(j)]}\rangle. \tag{4.21}$$

By (4.21) it is possible to see that D_F is not related to any quantity easily estimated from the Lyapunov numbers. Mori's estimate (4.16) follows from (4.21) by assuming only (4.19) and no fluctuations of the eigenvalues of the matrix A_{ij}.

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