Characterization of the Power Decays of Survival Probabilities at Long Times for Free Quantum Particle System

Manabu Miyamoto*

Department of Physics, Waseda University, Tokyo 169-8555, Japan

(Received February 4, 2003)

The power-law decrease of the survival probability $S(t)$ at long times for the one-dimensional free-particle system is shown to be characterized by the small-momentum behavior of the initial wave packet. We consider and compare the result based on the resolvent analysis and that on the asymptotic method of the Fourier integral, where the effect of a novelty of a one-dimensional free-resolvent and that of the discontinuity of initial states at zero momentum on the behavior of $S(t)$ are pointed out.

KEYWORDS: survival probability, power decay law, initial state

1. Introduction

The long-time behavior of the quantum systems involving the decaying or scattering process has been investigated over the years. One of the most disputable issues of the study concerns the power-law decrease of the survival probability $S(t)$ at long times. It is theoretically predicted for unstable systems\(^1,\)\(^2\) and for scattering potential systems\(^3\)–\(^8\) however, no experimental evidence for such a power law has been found.\(^9\) Thus, to clarify the observability of the power-law decrease, further investigations into the long-time behavior of $S(t)$ are required. In recent researches, another aspect in the power-law behavior at long times has been clarified for the one-dimensional free-particle system. For such a system, the wave packets, being initially Gaussian, decrease like $t^{-1/2}$ at long times, but this fact does not necessarily hold for another initial condition. Indeed, by choosing a spatial power-law wave packet at an initial time, the “anomalous decay” of its maximum can occur with the form $t^{-\alpha/2}$ ($1/2 < \alpha < 1$)\(^10,\)\(^11\) which is clearly slower than $t^{-1/2}$. Further investigation of $t^{-1/2}$ is also studied in relation to the dwell time\(^12\) and the time operator,\(^13,\)\(^14\) and is realized for the initial wave packets vanishing at zero momentum. However, as far as the author knows, a general relation between such various behaviors and initial states has not been well studied.

In this work, we confine ourselves to the long-time behavior of $S(t)$ for the one-dimensional free-particle system, but the consideration below is valid in an arbitrary dimension. We characterize the power-law decrease of $S(t)$, according to the small-momentum behavior of arbitrary initial wave packets. In particular, we consider and compare the result based on the resolvent analysis as used in refs. 3–5 with that on the asymptotic method of the Fourier integral.\(^15\) It is pointed out from the former approach that in the one-dimensional case, the free resolvent has a specialty determining the well-known $t^{-1/2}$ behavior of $S(t)$. From the latter one, how the discontinuity of initial wave packets at zero momentum affects the behavior of $S(t)$ is clarified.

For the one-dimensional free Hamiltonian $H_0 = -d^2/dx^2$, the solution of the Schrödinger equation, $\psi(x, t) = (e^{-itH_0}\psi)(x)$, reads

$$\psi(x, t) = (4\pi it)^{-1/2} \int_{\mathbb{R}} e^{i|x-y|^2/4t}\psi(y) dy,$$  \hspace{1cm} (1)

where $\psi(x)$ is the initial wave packet belonging to $L^2(\mathbb{R})$. The survival probability $S(t)$ of $\psi$ is defined by $S(t) := |A(t)|^2$, and $A(t)$ is the survival amplitude of $\psi$ given by

$$A(t) := \langle \psi, e^{-itH_0}\psi \rangle = \int_{\mathbb{R}} \overline{\psi(x, t)}\psi(x, t) dx \overset{2}{=} \int_{\mathbb{R}} |\psi(k)|^2 e^{-ik^2} dk. \hspace{1cm} (3)$$

The bar (‘’) denotes complex conjugate and $\psi(k)$ is the Fourier transform of $\psi(x)$. $S(t)$ is the probability that the state at a later time $t$ is found in the initial state.

2. Asymptotic Expansion of $e^{-itH_0}$

We first consider the asymptotic expansion of the time evolution operator $e^{-itH_0}$ at long times. The derivation is essentially the same as used for the short-range-potential systems.\(^4\) In the one-dimensional case, it reads\(^14\)

$$e^{-itH_0} = \sum_{j=0}^{n} (-1)^{j-1} \frac{\Gamma(j + 1/2)}{\pi^{1/2}(it)^{j+1/2}} G_{2j} + o(t^{-n-1/2}), \hspace{1cm} (4)$$

where $G_{2j}$’s are integral operators related to the free resolvent $R_0(z)$ [see eq. (6) below]. It is defined by

$$R_0(z)(\psi)(x) := ((H_0 - z)^{-1}\psi)(x) = -\frac{1}{2iz^{1/2}} \int_{\mathbb{R}} e^{iz^{1/2}|x-y|} \psi(y) dy,$$ \hspace{1cm} (5)

where $z$ is a complex number not in the spectrum of $H_0$, i.e., $[0, \infty)$, and $\text{Im } z^{1/2} > 0$ is chosen. Then, $R_0(z)$ is analytic everywhere on $\mathbb{C}\setminus[0, \infty)$. The derivation of eq. (4) is based on the estimation of asymptotic behavior of $R_0(z)$ at small and large $|z|$. $R_0(z)$ is expanded at small $|z|$ in the form

$$R_0(z) = \sum_{j=0}^{\infty} (iz^{1/2})^{-j-1} G_j. \hspace{1cm} (6)$$

* E-mail address: miyamo@hep.phys.waseda.ac.jp
where \( G_j \) is an integral operator defined by

\[
(G_j \psi)(x) := -\frac{1}{2z(j)} \int_{\mathbb{R}} |x-y|^2 \psi(y) \, dy.
\]

Then, eq. (4) is obtained by substituting the expansion (6) into the following formula

\[
e^{-\imath tH_0} = \frac{1}{2\pi} \int_0^\infty e^{-\imath \lambda} [R_0(\lambda + \imath 0) - R_0(\lambda - \imath 0)] \, d\lambda.
\]

Note that the right-hand side of the above may be regarded as a Cauchy's integral formula with the contour enclosing the spectrum of \( H_0 \). Furthermore, one see that

\[
R_0(\lambda + \imath 0) - R_0(\lambda - \imath 0) = 2\imath \sum_{j=0}^\infty (-1)^{j-1} \lambda^{-1/2} G_{2j},
\]

where \( G_j \) only appears for even \( j \). This leads to a result that asymptotic expansion (4) is composed of the terms of the half-power of \( t \). In addition, the dominating term of expansion (4) being \( t^{-1/2} \) is a direct consequence from the singular behavior of the one-dimensional free resolvent at small \( z \) [see r.h.s. of eq. (5)]. This fact is a special matter in one dimension. In fact, such a singularity does not appear for, e.g., a three-dimensional case, because the three-dimensional free resolvent has no singularity at \( z = 0^{+} \). This singularity may be also considered as a zero-energy resonance. Indeed, eq. (4) is rewritten in the form,

\[
e^{-\imath tH_0} = \pi^{1/2}(it)^{-1/2} \langle \varphi_0, \cdot \rangle \varphi_0 + o(t^{-1/2}),
\]

where \( \varphi_0(x) := (2\pi)^{-1/2} \) is the “zero-energy solution” of \( H_0 \varphi_0 = 0 \), and \( \langle \varphi_0, \cdot \rangle \) is the projection on \( \varphi_0 \). This is just the same form often found in the short-range-potential systems in one and three dimension as an “exceptional” case, where existence of the zero-energy solution changes the well-known decrease like \( t^{-3/2} \) into the slower one like \( t^{-1/2} \).

The equal signs in eqs. (4), (6), (8) and (9) are guaranteed only as those in the Banach space of the operators mapping \( L^{2,s}(\mathbb{R}) \) to \( L^{2,-s'}(\mathbb{R}) \), where \( s \) and \( s' \) are positive real numbers. \( L^{2,s}(\mathbb{R}) \) is the weighted \( L^2 \) space composed of the functions \( \psi(x) \) satisfying

\[
\int_{\mathbb{R}} (1 + x^2)^s \psi(x)^2 \, dx < \infty.
\]

Then it holds that \( L^{2,s}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset L^{2,-s'}(\mathbb{R}) \). In particular, this treatment enables us to extend \( R_0(z) \) to a real and positive \( z \), i.e., \( R_0(\lambda + \imath 0) \). However, it is significant to note that such a formulation restricts us to the initial states being in \( L^{2,s}(\mathbb{R}) \) with an appropriately large \( s > 0 \). This fact can be also recognized straightforwardly in the following manner: With respect to only eq. (4), one can rederive it without resort to the knowledge of \( R_0(z) \), expanding the exponential function of the integrand in eq. (1). Then, \( \psi(x, t) \) reads

\[
\psi(x, t) = \sum_{j=0}^n (-1)^{j-1} \frac{\Gamma(j + 1/2)}{\pi(j + 1/2)} (G_{2j} \psi)(x)
+ t^{-n-3/2} (K_n \psi)(x),
\]

where the remainder \( K_n \) stands for the integral operator

\[
(K_n \psi)(x) = \text{const.} \int_{\mathbb{R}} \int_0^1 \zeta^n e^{i|x-y|^2(1-\xi)/4\zeta} d\xi 
\times |x-y|^{2(n+1)} \psi(y) \, dy.
\]

However, in order to guarantee the termwise integration in the formal expansion (12), we have to assume \( \psi(x) \) to satisfy eq. (11) with \( s > 2n + 5/2 \). An integer \( n \) is chosen, according to the order which we require in the approximation. Indeed, this choice of \( \psi \) enables us to obtain

\[
\int_{\mathbb{R}} |x-y|^{2j} \psi(y) \, dy \leq 2^{2j} \int_{\mathbb{R}} |x-y|^{4j} (1 + y^2)^s \, dy \int_{\mathbb{R}} (1 + y^2)^s |\psi(y)|^2 \, dy < \infty
\]

for all \( x \in \mathbb{R} \) and for all \( j = 0, 1, \ldots, n+1 \), where we have used the Schwarz inequality. Then, \( (G_{2j} \psi)(x) \) for all \( j = 0, 1, \ldots, n \) and the remainder \( (K_n \psi)(x) \) are ensured to exist without divergence.

3. Various Long-Time Behaviors of \( A(t) \)

Substitution of eq. (4) [or (12)] into eq. (2) leads to asymptotic behaviors of \( A(t) \) at long times

\[
A(t) = \sum_{j=0}^n (-1)^{j-1} \frac{\Gamma(j + 1/2)}{\pi(j + 1/2)} \langle \psi, G_{2j} \psi \rangle + O(t^{-n-3/2}).
\]

At a glance, one may expect from expansion (14) that \( A(t) \) asymptotically shows the behavior like \( t^{-1/2} \). However, such an estimation is sometimes violated for those \( \psi \)'s for which \( \langle \psi, G_{2j} \psi \rangle \) vanishes with \( j \) up to some integer \( m - 1 \). To proceed the investigation of such an exceptional situation, we rewrite \( \langle \psi, G_{2j} \psi \rangle \) in terms of the initial wave packet \( \psi(k) \) in momentum representation. As in eq. (12), \( \psi(k) \) is expanded as

\[
\psi(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ikx} \psi(y) \, dy
\]

\[
= \sum_{j=0}^{2n+1} \frac{(-ik)^j}{\sqrt{2\pi} j!} \int_{\mathbb{R}} y^j \psi(y) \, dy + O(k^{2n+2}).
\]

The remainder is estimated by use of the following two inequalities for \( j = 2n + 2 \):

\[
\int_{\mathbb{R}} |y^j \psi(y) \, dy |^2 \leq \int_{\mathbb{R}} (1 + y^2)^{j-s} \, dy \int_{\mathbb{R}} (1 + y^2)^s |\psi(y)|^2 \, dy.
\]

Note that the right-hand side of the above includes the same integral as in eq. (11). This implies that the validity of expansion (15) is ensured by assumption (11) with \( s > 2n + 5/2 \). Expansion (15) leads to

\[
\psi^{(j)}(0) := \frac{d^j \psi(k)}{dk^j} \bigg|_{k=0} = \frac{(-i)^j}{\sqrt{2\pi}} \int_{\mathbb{R}} y^j \psi(y) \, dy.
\]
This enables us to obtain a desirable form of $\langle \psi, G_2 \psi \rangle$:

$$
\langle \psi, G_2 \psi \rangle = (-1)^{j-1} \pi \sum_{j'=0}^{2j} \frac{\psi^{(2j-j')}(0)\psi^{(j')}(0)}{(2j-j')!(j')!}.
$$  \tag{18}

Let us now consider such an initial wave packet $\psi$ that satisfies the condition

$$
\langle \psi, G_2 \psi \rangle = 0 \quad \text{for} \quad j = 0, 1, \ldots, m - 1,
$$  \tag{19}

where $m$ is an arbitrary positive integer not greater than $n$. One can then prove that condition (19) is actually equivalent to the following one:

$$
\hat{\psi}^{(j)}(0) = 0 \quad \text{for} \quad j = 0, 1, \ldots, m - 1.
$$  \tag{20}

This condition may be rewritten by

$$
\hat{\psi}(k) = O(k^m) \quad \text{as} \quad k \to 0.
$$  \tag{21}

Under the condition (20), inner product $\langle \psi, G_2 m \psi \rangle$ reads

$$
\langle \psi, G_2 m \psi \rangle = \frac{(-1)^{m-1}}{(m)!} \psi^{(m)}(0)^2,
$$  \tag{22}

which may remain. In short, if the initial state $\psi$ satisfies the condition (20), the leading term of asymptotic expansion (14) for $A(t)$ reads

$$
A(t) = \frac{\Gamma(m + 1/2)}{(m!)^2(it)^{m+1/2}}|\hat{\psi}^{(m)}(0)|^2 + O(t^{-m-3/2})
$$  \tag{23}

as $t \to \infty$, and $A(t)$ decreases asymptotically like $t^{-m-1/2}$. It is interesting to consider the special initial state $\psi$ whose differential coefficient $\hat{\psi}^{(j)}(0)$ vanishes for every $j$. Then, the above result tells us that, $A(t)$ decreases faster than any negative power of $t$, but slower than exponentials. An example of such a $\psi$ is constructed by use of $C_0^\infty$ functions.

As is pointed out after eq. (11), in our formulation, we have assumed the initial states being in $L^2(R)$ for an appropriate positive $s$. This assumption is seen to be at least satisfied by all rapidly decreasing functions, including Gaussian wave packets, with an arbitrary $s$. However, the assumption seems slighty strong, because it is sometimes not satisfied by the power-law-tail wave packet. Those wave packets often exhibit not only the divergence of $\hat{\psi}(0)$ but also a discontinuous behavior at zero momentum,

$$
\lim_{k \to 0} \hat{\psi}^{(m)}(k) \neq \lim_{k \to 0} \hat{\psi}^{(m)}(k).
$$  \tag{24}

Both cases can not occur under the condition (11) with $s > 2n + 5/2$ [see also the comments after eqs. (13) and (15)]. In those cases, we actually have a possibility to obtain another asymptotic behavior of $A(t)$ beyond the description by eq. (23).

4. Initial States with Discontinuous Derivatives

In order to estimate the long-time behavior of $A(t)$ for initial states $\psi$ with derivatives discontinuous at zero momentum, we here use the asymptotic expansion technique of the Fourier integral, expressing eq. (3) in the energy representation over the variable $E = k^2$,

$$
A(t) = \frac{1}{2} \sum_{\sigma = \pm} \int_0^\infty E^{-1/2} |\hat{\psi}^{(\sigma)}(1/2)|^2 e^{-itE} dE,
$$  \tag{25}

where $\sigma$ represents the degeneracy in the momentum. For our purpose, it is convenient to decompose $|\hat{\psi}(\sigma E^{1/2})|^2$ into the following forms:

$$
|\hat{\psi}(\pm E^{1/2})|^2 = \pm E^{1/2} \mathcal{O}_\pm(E) + \mathcal{E}_\pm(E),
$$  \tag{26}

where

$$
\mathcal{O}_\pm(E) = \sum_{r+s=\text{odd}} \frac{E^{(r+s+1)/2}}{r!s!} \bar{\psi}^{(r)}(0) \psi^{(s)}(0),
$$  \tag{27}

$$
\mathcal{E}_\pm(E) = \sum_{r+s=\text{even}} \frac{E^{(r+s)/2}}{r!s!} \bar{\psi}^{(r)}(0) \psi^{(s)}(0),
$$  \tag{28}

as $E \to 0$. The differentiability of $\hat{\psi}(k)$ has been assumed everywhere without the origin $k = 0$. Furthermore, we used the notations $\bar{\psi}^{(r)}(0) := \psi^{(r)}(k)$. It is important to note that as eq. (26) is inserted into eq. (25), $\mathcal{O}_\pm(E)$ does not cause a singularity at $E = 0$ but $\mathcal{E}_\pm(E)$ can cause it. This fact may demand a different treatment between $\mathcal{O}_\pm(E)$ and $\mathcal{E}_\pm(E)$, to derive an asymptotic expansion of eq. (25),

$$
A(t) \sim \frac{1}{2} \sum_{j=0} \sigma \mathcal{O}_\sigma^{(j)}(0) + \frac{\Gamma(j+1/2)}{j!(it)^{j+1/2}} \mathcal{E}_\sigma^{(j)}(0),
$$  \tag{29}

where it was assumed that both $\lim_{E \to \infty} \mathcal{O}_\pm^{(j)}(E) = 0$ and $\lim_{E \to \infty} \mathcal{E}_\pm^{(j)}(E) = 0$ hold. Equation (29) implies that if both $\mathcal{O}_\sigma^{(j)}(0)$ and $\mathcal{E}_\sigma^{(j)}(0)$ vanish with $j$ up to an integer $m$, $A(t)$ may asymptotically show the power-law decrease like $t^{-m-1/2}$ as a dominant one. This situation can be realized by the following condition, similar to the condition (20), i.e.,

$$
\hat{\psi}^{(j)}(0) = 0 \quad \text{for} \quad j = 0, 1, \ldots, m - 1.
$$  \tag{30}

In this case, eqs. (27) and (28) read

$$
\mathcal{O}_\pm(E) = \frac{2E^m}{m!} \text{Re} \left[ \frac{\bar{\psi}^{(m)}(0)\psi^{(m+1)}(0)}{m!(m+1)!} \right] + O(E^{m+1}),
$$  \tag{31}

$$
\mathcal{E}_\pm(E) = \frac{E^m}{(m!)^2} |\psi^{(m)}(0)|^2 + O(E^{m+1})
$$  \tag{32}

as $E \to 0$, respectively. Then, substituting them into eq. (29), we can obtain the expected result that

$$
A(t) \sim \frac{1}{2} \sum_{\sigma = \pm} \sigma \frac{2\text{Re} \left[ \bar{\psi}^{(m)}(0)\psi^{(m+1)}(0) \right]}{(m!)^{3/2}(it)^{m+1/2}} + \frac{\Gamma(m+1/2)|\psi^{(m)}(0)|^2}{(m!)^{3/2}(it)^{m+1/2}} + O(t^{-m-3/2}).
$$  \tag{33}

It is worth noting that the first term on the right-hand side of the above can vanish after summing over $\sigma = \pm$, if $\hat{\psi}^{(m)}(k)$ is continuous at the zero momentum, i.e.,
\( \hat{\psi}^{(m)}(-0) = \hat{\psi}^{(m)}(+0) = [\hat{\psi}^{(m)}(0)]. \) In such a case, as is expected, \( A(t) \) in eq. (33) just reads eq. (23). On the other hand, we can obtain the formula (33) with a nonvanishing first term for the initial wave packets considered in ref. 12, e.g.,

\[
\psi(x) := N(x + i\alpha)^{-2},
\]

where \( \alpha \) is real and positive, and \( N = \sqrt{2\alpha^{3/\pi}} \) being the normalization constant. The Fourier transform of the \( \psi(x) \) becomes \( \hat{\psi}(k) = -2\alpha^{3/2}\Theta(k)ke^{-\alpha k}, \) where \( \Theta(k) \) is the Heaviside function taking value 0 or 1 for \( k < 0 \) or \( k > 0 \), respectively. Thus \( \psi(k) \) has a property not only (20) [or (30)] but also (24) with \( m = 1 \). Then eq. (33) reads

\[
A(t) \sim \frac{\Gamma(3/2)^{2}}{2[m!]^{2}(it)^{3/2}} + O(t^{-2}).
\]

It is important to see that a discontinuous behavior of \( \hat{\psi}^{(1)}(k) \) at zero momentum makes the next leading term of \( A(t) \) being slower, i.e., \( O(t^{-2}) \) unlike \( O(t^{-5/2}) \) in the continuous case [see eq. (23)]. This may cause a significant effect on an experimental observation of the power-law decrease, because for such a discontinuous \( \psi \) the onset of the power law may be late. We remark that, as it was shown that \( \psi(0,t) \sim t^{-1/2} \), the above result tells us that the fact \( \psi(0,t) \sim t^{-1} \) does not necessarily mean \( A(t) \sim t^{-1} \). Such a disagreement was studied in ref. 18.

### 5. An Implication for the Potential Systems

Our analysis for the free particle system predicts at least the existence of the various power decreases of \( A(t) \) for potential systems, through the wave operators:\[19\]

\[ W_{\pm} := \lim_{\epsilon \to \pm 0} e^{itH_{0}}e^{-itH_{0}} \]

where \( H = H_{0} + V \) is the total Hamiltonian and \( V \) the potential. We also assume that potential \( V \) belongs to \( L^{1}(R) \cap L^{2}(R) \). Then, \( W_{\pm} \) are guaranteed to have an important property

\[
e^{-itH}P_{ac} = W_{\pm}e^{-itH_{0}}W_{\pm}^{*},
\]

where the asterisk (*) denotes adjoint operator. \( P_{ac} \) is the projection operator with respect to the absolutely continuous part of \( H \), i.e., scattering states for \( H \). They also have the property that \( W_{\pm}W_{\pm}^{*} = P_{ac} \) and \( W_{\pm}W_{\pm}^{*} = I \) with \( I \) the identity operator. In addition, if \( V \) satisfies \( 0 \leq V(x) \leq \text{const.} \), \( \infty \), \( W_{\pm} \), and \( P_{ac} \) are unitary and identity operators, respectively. It should be noted that, for a state \( \psi(x) \) with condition (20), eqs. (23) and (36) assert that the survival amplitude of “initial states” \( W_{\pm} \) for potential systems behave like

\[
\langle W_{\pm}\psi, e^{-itH}W_{\pm}\psi \rangle \sim \frac{\Gamma(m+1/2)}{(m!)^{2}(it)^{m+1/2}}|\hat{\psi}^{(m)}(0)|^{2}.
\]

This implies that, in principle, there exist the initial states to cause the various power decreases of the survival amplitude for potential systems. However how such initial states are explicitly specified is unclear.

### 6. Concluding Remarks

We have shown how the various power decreases of the survival amplitude \( A(t) \) [and thus the survival probability \( S(t) \) for the one-dimensional free-particle system can appear, according to the small-momentum behaviors of initial states. It is also seen from this result that such various behaviors of \( A(t) \), other than the well known \( t^{-3/2} \) \( \sim 3 \) are realizable for the short-range-potential systems. The two approaches based on the asymptotic expansion of the free resonant and that of the Fourier integral are concerned. In particular, the latter one enables us to deal with the initial states with derivatives discontinuous at zero momentum, for which \( A(t) \) shows a different behavior from that for the initial states with continuous derivatives. This implies that, for initial states with such a discontinuous property, the difference between \( R_{0}(\lambda \pm 0) \) may have another form than eq. (9). Since the analyses in refs. 3–5 were based on eq. (9), the asymptotic expansion of \( e^{-itH} \) derived there may be changed for such initial states. The latter approach can be also applied to the potential systems,\[20\] but analytic solutions of the time-independent Schrödinger equation is needed unlike the results in refs. 3–5. Asymptotic expansion of \( (e^{-itH}\psi(x)) \) at long times for the short-range-potential systems was also attained by several authors.\[6–8\] By any method, our analysis herein could be extended to the short-range-potential systems.

### Acknowledgments

The author would like to express his gratitude to the organizers of the Waseda International Symposium on Fundamental Physics 2002. He also would like to thank Professor I. Oiba and Professor H. Nakazato for useful and helpful discussions.